A COMPREHENSIVE MODERN TREATMENT
OF ANALYTIC FUNCTION THEORY
AND ITS RAMIFICATIONS

THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

VOLUME II

A. I. MARKUSHEVICH

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Richard A. Silverman, *Editor*

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THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

Volume II

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Revised English Edition
Translated and Edited by

Richard A. Silverman

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AUTHOR'S PREFACE

The theory of analytic functions of a complex variable is one of the outstanding accomplishments of classical mathematics. The rudiments of the theory had already begun to emerge in the eighteenth century, in the investigations of Euler and d'Alembert, but the full flowering of the subject was only possible in the nineteenth century, as a result of the fundamental researches of Cauchy, Riemann and Weierstrass. Its methods are widely used in many branches of mathematics, ranging from number theory to fluid dynamics, elasticity theory and electrodynamics. Even in our own day, when young scientists are so often preoccupied with more abstract branches of mathematics, the theory of analytic functions remains as significant as ever, both as a treasury of flexible and powerful analytic tools, and as a prototype or model of other theories dealing with generalizations of analytic functions in various directions.

It is the author's pleasant duty to cordially thank Dr. Richard A. Silverman, who translated the book into English, introduced many appropriate changes making the book easier to read, and enhanced the English-language edition with an abundance of problems and exercises.

This book stems from lectures given over the course of many years at Moscow University. In commending it to the attention of its new readership, the author would consider his purpose fulfilled if the present edition served to stimulate the interest of young mathematicians in one of the most remarkable branches of exact science, created by two centuries of joint effort on the part of

outstanding scholars of many nations, and if, at the same time, it represented a contribution by him and the translator towards the strengthening of cultural and scientific cooperation between their two countries.

A. I. M.

TRANSLATOR'S PREFACE

It would be difficult to recall in detail the ways in which the present version departs from the Russian original. Suffice it to say that in keeping with my philosophy of "enlightened translation," I thought it entirely appropriate to interact strongly with the initial text, especially since it lies within the area of my mathematical competence. This has led to a redistribution of the subject matter over three English volumes (the Russian original contains some 700 oversized pages, many in very small print), an overall restyling both of the typography and the art work (in particular, more theorems, examples, remarks, etc. have been explicitly identified as such), an adaptation of the bibliographic materials to suit the needs of those who do not know Russian (a class which presumably includes almost all the readers of this series), and so on. Certainly the most significant change is the addition of nearly 700 relevant problems, closely paralleling the text and stemming from a variety of sources, the most important of which is the excellent collection by Volkovyski, Lunts and Aramanovich (Moscow, 1961). The entire manuscript has also been read by Dr. T. J. Rivlin, who made many helpful suggestions.

Sections marked with asterisks can be omitted on first reading. However, it is only fair to say that they have been so marked merely because of their relative difficulty, and actually contain some of the most interesting material in the book.

R. A. S.

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THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

Volume II

Part 1

LAURENT SERIES CALCULUS OF RESIDUES

1

LAURENT SERIES. ISOLATED SINGULAR POINTS

1. Laurent's Theorem

In Volume I of this course,[1] we devoted considerable attention to the subject of power series, i.e., expansions in *nonnegative* integral powers of the variable $z - z_0$. We now consider series of a related type, which involve *arbitrary* integral powers of $z - z_0$. These series closely resemble power series both in structure and basic properties. Our first result is the analogue of the Cauchy-Hadamard theorem (Theorem I.16.1):[2]

THEOREM 1.1. *Given a series*

$$A_0 + A_1(z - z_0)^{-1} + A_2(z - z_0)^{-2} + \cdots + A_n(z - z_0)^{-n} + \cdots \quad (1.1)$$

of negative powers of $z - z_0$, *let*

$$r = \overline{\lim_{n \to \infty}} \sqrt[n]{|A_n|},$$

and let γ *be the circle* $|z - z_0| = r$, *with interior* $I(\gamma)$ *and exterior* $E(\gamma)$.

---

[1] We refer to A. I. Markushevich, *Theory of Functions of a Complex Variable, Volume I* (translated by R. A. Silverman), Prentice-Hall, Inc., Englewood Cliffs, N.J. (1965). This book will henceforth be called simply *Volume I.*

[2] Theorems, chapters, sections, formulas, problems, figures and page numbers cited from Volume I will be indicated by the prefix I. Thus, by Theorem I.16.1, Section I.84 and formula (I.16.54), we mean Theorem 16.1, Section 84 and formula (16.54) of Volume I, and so on.

Then there are three possibilities:

1. *If* $r = 0$, *the series is absolutely convergent for all* z *in the extended plane except* $z = z_0$;
2. *If* $0 < r < \infty$, *the series is absolutely convergent for all* $z \in E(\gamma)$ *and divergent for all* $z \in I(\gamma)$;
3. *If* $r = \infty$, *the series is divergent for all finite* z.

Proof. The substitution

$$Z = \frac{1}{z - z_0} \tag{1.2}$$

converts (1.1) into the power series

$$A_0 + A_1 Z + A_2 Z^2 + \cdots + A_n Z^n + \cdots,$$

with radius of convergence

$$\frac{1}{\overline{\lim\limits_{n \to \infty}} \sqrt[n]{|A_n|}} = \frac{1}{r}.$$

The rest of the proof is an immediate consequence of Theorem I.16.1, and the fact that (1.2) transforms the points $z = z_0$, $z = \infty$ into the points $Z = \infty$, $Z = 0$.

Remark 1. Obviously, the behavior of the series (1.1) for $r = 0$ and $r = \infty$ can be regarded as the appropriate limiting case of its behavior for $0 < r < \infty$. We shall henceforth assume that $r < \infty$, so that (1.1) converges on a domain, i.e., the exterior of the circle $\gamma: |z - z_0| = r$.

Remark 2. It is clear that every compact subset of $E(\gamma)$ is mapped into a compact subset of $I(\gamma)$ by the transformation

$$z = z_0 + \frac{1}{Z},$$

which is the inverse of the transformation (1.2). Therefore, according to Theorem I.16.3, the series (1.1) is uniformly convergent on every compact subset of $E(\gamma)$. It follows from Theorem I.15.8 that (1.1) represents a function $f(z)$ which is analytic at every finite point of $E(\gamma)$.

Remark 3. The function $f(z)$ is also analytic at $z = \infty$ (see p. I.125), since the function $f^*(\zeta) = f(1/\zeta)$ is analytic at $\zeta = 0$ (see Problem 1.1). Moreover $f(\infty) = A_0$, since

$$\lim_{z \to \infty} f(z) = A_0$$

(see p. I.86).

We now introduce series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \tag{1.3}$$

interpreted as the sum of the two series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n, \quad \sum_{m=1}^{\infty} a_{-m}(z - z_0)^{-m}. \tag{1.4}$$

The series (1.3) is known as a *Laurent series*, and is regarded as convergent if and only if both series (1.4) converge. In other words,

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \lim_{\nu \to \infty} \sum_{n=0}^{\nu} a_n(z - z_0)^n + \lim_{\mu \to \infty} \sum_{m=1}^{\mu} a_{-m}(z - z_0)^{-m},$$

or equivalently,

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \lim_{\substack{\mu \to \infty \\ \nu \to \infty}} \sum_{n=-\mu}^{\nu} a_n(z - z_0)^n, \tag{1.5}$$

where μ and ν approach infinity independently. Thus (1.5) means that given any $\varepsilon > 0$, there is an integer $N(\varepsilon) > 0$ such that

$$\left| \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n - \sum_{n=-\mu}^{\nu} a_n(z - z_0)^n \right| < \varepsilon$$

whenever $\mu > N(\varepsilon)$ and $\nu > N(\varepsilon)$.

According to this definition, the convergence behavior of a Laurent series depends on the corresponding behavior of the series (1.4). Consider the two circles $\Gamma: |z - z_0| = 1/R$ and $\gamma: |z - z_0| = r$, where

$$\frac{1}{R} = \overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|}, \qquad r = \overline{\lim_{m \to \infty}} \sqrt[m]{|a_{-m}|}.$$

Then the first of the series (1.4) converges absolutely and uniformly on every compact subset of $I(\Gamma)$ and diverges on $E(\Gamma)$, whereas the second of the series (1.4) converges absolutely and uniformly on every compact subset of $E(\gamma)$ and diverges on $I(\gamma)$. The domains $I(\Gamma)$ and $E(\gamma)$ have a nonempty intersection if and only if $r < R$, and if this condition is satisfied, the domain $D = I(\Gamma) \cap E(\gamma)$ is a ring-shaped region or *annulus*, consisting of all points z such that

$$r < |z - z_0| < R.$$

Clearly, both series (1.4) converge absolutely and uniformly on every compact subset of D, and hence the same is true of the series (1.3). It follows from Theorem I.15.8 that the function $f(z)$ defined by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \qquad (r < |z - z_0| < R) \tag{1.6}$$

is analytic on D. Moreover, since one or the other of the series (1.4) diverges on the exterior of D, so does the series (1.3).

Remark. Henceforth, in talking about a Laurent series, we shall always assume that the condition $r < R$ is satisfied, so that there is an annulus on which the series converges.

THEOREM 1.2. *The coefficients of the Laurent series* (1.6) *are given by the formula*

$$a_k = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}}\, dz \qquad (k = 0, \pm 1, \pm 2, \ldots), \tag{1.7}$$

where γ_ρ *is any circle* $|z - z_0| = \rho$, $r < \rho < R$.

Proof. The series (1.6) is uniformly convergent on γ_ρ, and hence the same is true of the series obtained by multiplying (1.6) by

$$\frac{1}{2\pi i}\frac{1}{(z - z_0)^{k+1}},$$

where k is an arbitrary integer. It follows from Theorem I.15.4 that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}}\, dz &= \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi i}\int_{\gamma_\rho} (z - z_0)^{n-k-1}\, dz \\ &= \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi}\int_0^{2\pi} \rho^{n-k} e^{i(n-k)\theta}\, d\theta = a_k,\end{aligned}$$

where we use the fact that the last integral on the right vanishes if $n \neq k$ and equals 2π if $n = k$.

COROLLARY 1. *Let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \qquad \varphi(z) = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n \tag{1.8}$$

be two Laurent series, which converge on annuli D *and* Δ, *respectively, and suppose* D *and* Δ *contain the same circle* $\gamma: |z - z_0| = \rho$. *Then*

$$a_k = b_k \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

i.e., the two series (1.8) *are identical.*

Proof.

$$a_k = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}}\, dz = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{k+1}}\, dz = b_k \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

COROLLARY 2. *There is a unique Laurent series in the variable* $z - z_0$ *which has a given sum on an annulus* $D: r < |z - z_0| < R$.

Proof. Choose $D = \Delta$ in Corollary 1.

Our next result is the analogue of Cauchy's theorem on the expansion of an analytic function in power series (Theorem I.16.7).

THEOREM 1.3 (*Laurent's theorem*). *Let $f(z)$ be an analytic function on an annulus $D: r < |z - z_0| < R$.*[3] *Then there exists a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \tag{1.9}$$

converging to $f(z)$ on D.

Proof. Let z_1 be an arbitrary point of D, and choose values of r' and R' satisfying the inequalities

$$r < r' < |z_1 - z_0| < R' < R.$$

Then the annulus

$$D': r' < |z - z_0| < R'$$

is contained in D and contains the point z_1. Let γ_ρ be the circle $|z - z_0| = \rho$, $r < \rho < R$, and let Γ be a circle with center z_1 such that $\Gamma \subset D'$ (see Figure 1.1). Since the function

$$\varphi(z) = \frac{f(z)}{z - z_1}$$

is analytic on $D - \{z_1\}$, it follows from Cauchy's integral theorem for a system of contours (Theorem I.13.4) that[4]

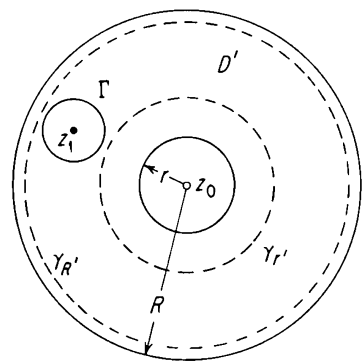

FIGURE 1.1

$$\frac{1}{2\pi i}\int_{\gamma_{R'}} \frac{f(z)}{z - z_1}\,dz = \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(z)}{z - z_1}\,dz + \frac{1}{2\pi i}\int_{\Gamma} \frac{f(z)}{z - z_1}\,dz. \tag{1.10}$$

The last integral on the right is a Cauchy integral, and equals $f(z_1)$. Therefore (1.10) becomes

$$f(z_1) = \frac{1}{2\pi i}\int_{\gamma_{R'}} \frac{f(z)}{z - z_1}\,dz - \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(z)}{z - z_1}\,dz,$$

or

$$f(z_1) = \frac{1}{2\pi i}\int_{\gamma_{R'}} \frac{f(z)}{z - z_1}\,dz + \frac{1}{2\pi i}\int_{\gamma_{r'}} \frac{f(z)}{z_1 - z}\,dz. \tag{1.11}$$

---

[3] The annulus may reduce to a "deleted neighborhood", i.e., to a disk without its center (if $r = 0$, $R < \infty$), to the exterior of a circle with the point at infinity deleted (if $r > 0$, $R = \infty$), or to the whole plane with the points z_0 and ∞ deleted (if $r = 0$, $R = \infty$). See the remark on p. 10.

[4] The circles $\gamma_{R'}$, $\gamma_{r'}$ and Γ are all traversed in the same (counterclockwise) direction.

As we now show, the first integral on the right in (1.11) leads to the nonnegative powers of $z - z_0$ in the Laurent series (1.9), while the second integral leads to the negative powers of $z - z_0$. With this in mind, we express the factor

$$\frac{1}{z - z_1} \qquad (z \in \gamma_{R'})$$

appearing in the first integral as a geometric series with ratio

$$\frac{z_1 - z_0}{z - z_0},$$

where

$$\left|\frac{z_1 - z_0}{z - z_0}\right| = \frac{|z_1 - z_0|}{R'} = \theta < 1.$$

The result is

$$\begin{aligned} \frac{1}{z - z_1} &= \frac{1}{(z - z_0) - (z_1 - z_0)} \\ &= \frac{1}{z - z_0} \frac{1}{1 - \dfrac{z_1 - z_0}{z - z_0}} = \sum_{n=0}^{\infty} \frac{(z_1 - z_0)^n}{(z - z_0)^{n+1}} \end{aligned} \tag{1.12}$$

[cf. (I.16.54)], where the series on the right is uniformly convergent on $\gamma_{R'}$, since

$$\left|\frac{(z_1 - z_0)^n}{(z - z_0)^{n+1}}\right| = \frac{\theta^n}{R'} \qquad (0 < \theta < 1).$$

Therefore the series

$$\frac{1}{2\pi i} \frac{f(z)}{z - z_1} = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \frac{f(z)}{(z - z_0)^{n+1}} (z_1 - z_0)^n, \tag{1.13}$$

obtained by multiplying (1.12) by the function

$$\frac{1}{2\pi i} f(z)$$

(which is bounded on $\gamma_{R'}$) is uniformly convergent on $\gamma_{R'}$, and we can integrate (1.13) term by term along $\gamma_{R'}$, obtaining

$$\frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z - z_1} dz = \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n, \tag{1.14}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma_{R'}} \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad (n = 0, 1, 2, \ldots). \tag{1.15}$$

Next we again consider (1.11) and express the factor

$$\frac{1}{z_1 - z} \qquad (z \in \gamma_{r'})$$

appearing in the second integral as a geometric series with ratio

$$\frac{z - z_0}{z_1 - z_0},$$

where

$$\left|\frac{z - z_0}{z_1 - z_0}\right| = \frac{r'}{|z_1 - z_0|} = \vartheta < 1.$$

The result is

$$\frac{1}{z_1 - z} = \frac{1}{(z_1 - z_0) - (z - z_0)} = \frac{1}{z_1 - z_0} \frac{1}{1 - \dfrac{z - z_0}{z_1 - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z_1 - z_0)^{n+1}}.$$

The series on the right is uniformly convergent on $\gamma_{r'}$, since

$$\left|\frac{(z - z_0)^n}{(z_1 - z_0)^{n+1}}\right| = r'\vartheta^n \qquad (0 < \vartheta < 1),$$

and hence so is the series

$$\begin{aligned} \frac{1}{2\pi i} \frac{f(z)}{z - z_1} &= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \frac{f(z)}{(z - z_0)^{-n}} (z_1 - z_0)^{-n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \frac{f(z)}{(z - z_0)^{-n+1}} (z_1 - z_0)^{-n}. \end{aligned} \tag{1.16}$$

Integrating (1.16) term by term along $\gamma_{r'}$, we find that

$$\frac{1}{2\pi i} \int_{\gamma_{r'}} \frac{f(z)}{z - z_1}\, dz = \sum_{n=1}^{\infty} a_{-n}(z_1 - z_0)^{-n}, \tag{1.17}$$

where

$$a_{-n} = \frac{1}{2\pi i} \int_{\gamma_{r'}} \frac{f(z)}{(z - z_0)^{-n+1}}\, dz \qquad (n = 1, 2, \ldots). \tag{1.18}$$

Substituting (1.14) and (1.17) into (1.11), and replacing z_1 by z (z_1 is arbitrary), we finally find that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n,$$

which is the desired Laurent series (1.9). Moreover, using Theorem I.13.4 to replace $\gamma_{r'}$ and $\gamma_{R'}$ by γ_ρ ($r < \rho < R$), we can combine (1.15) and (1.18) into a single formula

$$a_n = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

in complete agreement with (1.7).

Remark. If $r = 0$ and $R < \infty$, the annulus figuring in Theorem 1.3 reduces to a disk without its center. Such a set $\mathcal{N}(z_0) - \{z_0\}$ is called a *deleted neighborhood of* z_0, and will often be denoted by $\mathcal{N}'(z_0)$. A function $f(z)$ is said to have a *Laurent expansion* at $z = z_0$ if $f(z)$ has a Laurent expansion (in the variable $z - z_0$) in some deleted neighborhood of z_0. Similarly, if $r > 0$ and $R = \infty$, the annulus reduces to the exterior of a circle $\gamma: |z - z_0| = r$, with the point at infinity deleted. Such a set $E(\gamma) - \{\infty\} = \mathcal{N}'(\infty)$ is called a *deleted neighborhood of* ∞. A function $f(z)$ is said to have a *Laurent expansion at* $z = \infty$ if $f(z)$ has a Laurent expansion in some deleted neighborhood of ∞ (see Problem 1.4).

Example. In the annulus $0 < |z| < 1$, and hence at $z = 0$, the function

$$f(z) = \frac{1}{z(1 - z)}$$

has the Laurent expansion

$$\frac{1}{z} + \sum_{n=0}^{\infty} z^n,$$

while in the annulus $0 < |z - 1| < 1$, and hence at $z = 1$, $f(z)$ has the Laurent expansion

$$-\frac{1}{z - 1} + \sum_{n=0}^{\infty} (-1)^n (z - 1)^n.$$

In the domain $1 < |z| < \infty$, and hence at $z = \infty$, $f(z)$ has the Laurent expansion

$$-\sum_{n=2}^{\infty} \frac{1}{z^n}.$$

The last expansion contains no positive powers of z, and is therefore valid at $z = \infty$, where $f(z)$ vanishes (see Remark 3, p. 4).

2. Poles and Essential Singular Points

Let $f(z)$ be a single-valued function which is analytic in a neighborhood of a point z_0, with the possible exception of the point z_0 itself, i.e., in a deleted neighborhood of z_0. This means that $f(z)$ is analytic on some domain

$$D: 0 < |z - z_0| < R.$$

If z_1 is a point of D satisfying the condition

$$0 < |z_1 - z_0| \leqslant \frac{R}{2},$$

then $f(z)$ is analytic on the disk

$$K_{z_1}: |z - z_1| < |z_0 - z_1|,$$

and hence $\{K_{z_1}, f(z)\}$ is a circular element, in the sense of Sec. I.84.[5] All the points of the boundary of K_{z_1}, i.e., of the circle

$$\Gamma_{z_1}: |z - z_1| = |z_0 - z_1|,$$

are regular points of this element, with the possible exception of the point z_0. Let $z_2 \neq z_1$ be any other point such that

$$0 < |z_2 - z_0| \leqslant \frac{R}{2}.$$

Then $f(z)$ is also analytic on the disk

$$K_{z_2}: |z - z_2| < |z_0 - z_2|,$$

and $\{K_{z_2}, f(z)\}$ is another circular element. As before, all the points of the circle

$$\Gamma_{z_2}: |z - z_2| = |z_0 - z_2|$$

are regular points of $\{K_{z_2}, f(z)\}$, with the possible exception of z_0. The behavior of all such circular elements of $f(z)$ with the point z_0 on their boundaries is clarified by the following

THEOREM 1.4. *If $f(z)$ is a single-valued analytic function in a deleted neighborhood $D: 0 < |z - z_0| < R$ of a point z_0, then z_0 is a regular point of the element $\{K_{z_1}, f(z)\}$ if and only if it is a regular point of every other element $\{K_{z_2}, f(z)\}$.*

Proof. If z_0 is a regular point of $\{K_{z_1}, f(z)\}$, then all the points of the circle Γ_{z_1} (the boundary of K_{z_1}) are regular points of $\{K_{z_1}, f(z)\}$. Therefore, according to Theorem I.17.10, the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!} (z - z_1)^n \tag{1.19}$$

converges on a disk $K_{z_1}^*: |z - z_1| < \rho$ such that $\rho > |z_1 - z_0|$, i.e., such that z_0 is an interior point of $K_{z_1}^*$. Let $\varphi(z)$ be the sum of the series (1.19). Since $\varphi(z) = f(z)$ for all $z \in K_{z_1}$, it follows from the

[5] It is natural to call any (circular) element of the form $\{K, f(z)\}$ an element *of the function* $f(z)$, although this qualifying phrase was deliberately avoided in Volume I.

interior uniqueness theorem (Theorem I.17.1) that $\varphi(z) = f(z)$ for all $z \in K_{z_1}^* \cap D$. In particular, $\varphi(z) = f(z)$ for all z in some deleted neighborhood of z_0, i.e., for all $z \in \mathcal{N}'(z_0)$. [Note that $f(z)$ is as yet undefined at z_0.] Now consider any other element $\{K_{z_2}, f(z)\}$, such that $z_0 \in \Gamma_{z_2}$, where Γ_{z_2} is the boundary of K_{z_2}. Since, as just shown, $\varphi(z) = f(z)$ for all $z \in \mathcal{N}(z_0) \cap K_{z_2} \subset \mathcal{N}'(z_0)$, it follows that z_0 is a regular point of $\{K_{z_2}, f(z)\}$ [cf. p. I.382].

Conversely, suppose z_0 is a singular point of an element $\{K_{z_1}, f(z)\}$. Then z_0 must be a singular point of any other element $\{K_{z_2}, f(z)\}$, since if z_0 were a regular point of $\{K_{z_2}, f(z)\}$, the argument just given would imply that z_0 is a regular point of $\{K_{z_1}, f(z)\}$, contrary to hypothesis. This completes the proof.

Remark 1. Thus, under the hypotheses of the theorem, if z_0 is a regular point of one element of $f(z)$, it is a regular point of all other elements of $f(z)$.[6] In this case, we say that z_0 is a *regular point of* $f(z)$, and we complete the definition of $f(z)$ by setting $f(z_0) = \varphi(z_0)$, where $\varphi(z)$ is the sum of the Taylor series (1.19), thereby obtaining a function $f(z)$ which is analytic at z_0. Moreover, let $z_1 \neq z_0$ be any point at which $f(z)$, as originally defined, is analytic. Then z_1 is a regular point in the sense of this definition, since obviously z_1 is a regular point of every element of $f(z)$ with z_1 on its boundary. A regular point at which $f(z)$ is originally undefined is often called a *removable singular point.*

Remark 2. Similarly, under the same conditions, if z_0 is a singular point of one element of $f(z)$ with z_0 on its boundary, it is a singular point of all such elements. In this case, we say that z_0 is a *singular point of* $f(z)$, more precisely, an *isolated singular point (of single-valued character).* More generally, any point which is not a regular point of $f(z)$ will be called a singular point of $f(z)$, whether isolated or not. The term *singularity* is often used as a synonym for *singular point.*

Suppose $f(z)$ is analytic in a deleted neighborhood $D: 0 < |z - z_0| < R$ of an isolated singular point z_0. Then the basic tool for studying $f(z)$ is its Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (z \in D), \tag{1.20}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{1.21}$$

[6] Note that this conclusion relies on the fact that z_0 is an *isolated* point of the boundary of D. As shown by Example 2, p. I.393, under weaker conditions, it is perfectly possible for z_0 to be a regular point of one element of $f(z)$ and a singular point of another element.

and γ_ρ is the circle $|z - z_0| = \rho$ $(0 < \rho < R)$. The expansion (1.20) can also be used in the case where z_0 is a regular point of $f(z)$. Then $a_n = 0$ if $n < 0$, and the Laurent series reduces to a Taylor series.

THEOREM 1.5. *Let $f(z)$ be a single-valued analytic function defined on the domain $D: 0 < |z - z_0| < R$. Then z_0 is a regular point of $f(z)$ if and only if $f(z)$ is bounded in some deleted neighborhood of z_0.*

Proof. Suppose z_0 is a regular point of $f(z)$. Then there is a neighborhood $\mathcal{N}(z_0)$ and a function $\varphi(z)$ analytic on $\mathcal{N}(z_0)$ such that $f(z) = \varphi(z)$ for all $z \in \mathcal{N}'(z_0) = \mathcal{N}(z_0) - \{z_0\}$. Moreover, since $\varphi(z)$ is continuous on $\mathcal{N}(z_0)$, there is a finite constant $M > 0$ such that $|f(z)| = |\varphi(z)| \leqslant M$ for all $z \in \mathcal{N}'(z_0)$. In fact, setting $f(z_0) = \varphi(z_0)$, we have $|f(z)| \leqslant M$ for all $z \in \mathcal{N}(z_0)$.

Conversely, suppose that there is a deleted neighborhood $\mathcal{N}'(z_0) \subset D$ and a finite constant $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in \mathcal{N}'(z_0)$. Choosing a circle $\gamma_\rho: |z - z_0| = \rho$ $(0 < \rho < R)$ such that $\gamma_\rho \subset \mathcal{N}'(z_0)$, we use (1.21) to estimate the coefficients a_n of the Laurent series (1.20):

$$|a_n| \leqslant \frac{1}{2\pi} M \frac{2\pi}{\rho^{n+1}}, \quad \text{i.e.,} \quad |a_n| \leqslant \frac{M}{\rho^n}.$$

If $n < 0$, then, letting ρ $(0 < \rho < R)$ approach zero, we find that

$$a_n = 0 \qquad (n = -1, -2, \ldots).$$

Therefore the Laurent series (1.20) reduces to the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The sum of this series is analytic on the whole disk $|z - z_0| < R$ and coincides with $f(z)$ on D. Thus z_0 is a regular point of $f(z)$, as asserted.

COROLLARY. *The point z_0 is a singular point of $f(z)$ if and only if $f(z)$ is unbounded in every deleted neighborhood $\mathcal{N}'(z_0)$, i.e.,*

$$\overline{\lim_{z \to z_0}} |f(z)| = \infty.$$

According to the corollary, *a priori* we have the following two possibilities for the behavior of $f(z)$ in a neighborhood of an isolated singular point:

1. $f(z) \to \infty$ as $z \to z_0$;
2. $f(z)$ approaches no limit (finite or infinite) as $z \to z_0$.

Simple examples show that both cases are possible. To illustrate the first case, let

$$f(z) = \frac{1}{(z - z_0)^n}, \tag{1.22}$$

where n is a positive integer. Then obviously $f(z)$ is analytic for $|z - z_0| > 0$, and

$$\lim_{z \to z_0} f(z) = \infty.$$

To illustrate the second possibility, let

$$f(z) = \exp\left(\frac{1}{z - z_0}\right).$$

This function is also analytic for $|z - z_0| > 0$, but unlike (1.22), $f(z)$ approaches no limit as $z \to z_0$. In fact, let z lie on the straight line passing through z_0 parallel to the real axis, so that $z - z_0 = x - x_0$, where $x - x_0$ is real. Then

$$\exp\left(\frac{1}{x - x_0}\right) \to \infty \qquad (x > x_0)$$

as $x \to x_0$ from the right, whereas

$$\exp\left(\frac{1}{x - x_0}\right) \to 0 \qquad (x < x_0)$$

as $x \to x_0$ from the left. This behavior is shown in Figure 1.2, where we plot the modular surface of the function $\exp(1/z)$ [corresponding to the case

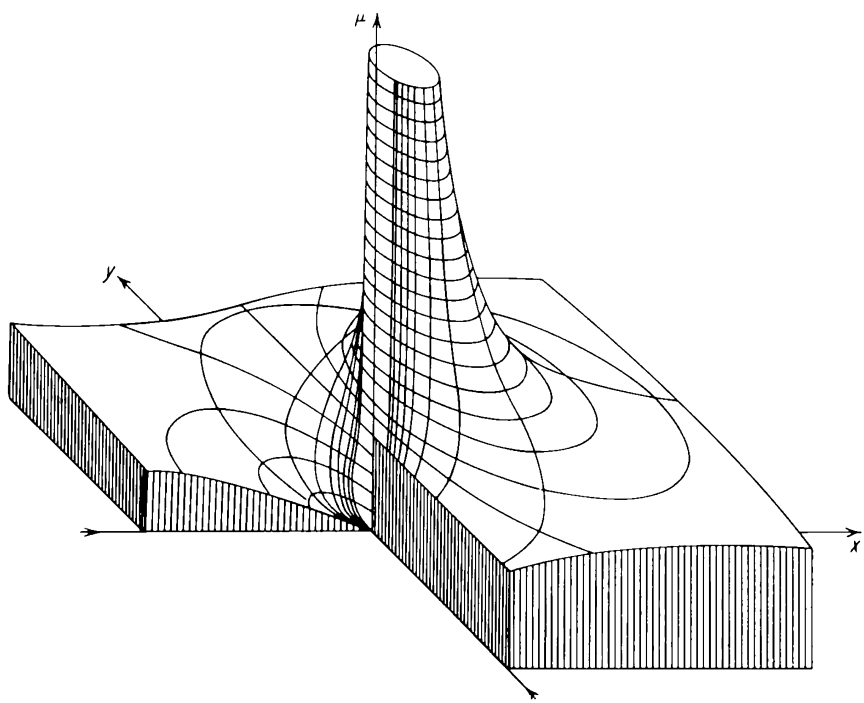

FIGURE 1.2

$z_0 = 0$], i.e., the surface $\mu = |\exp(1/z)|$. The curves $\mu = \text{const}$ encircle the "spike," while the curves $\arg \exp(1/z) = \text{const}$ go up the spike.

DEFINITION 1. *An isolated singular point z_0 of a function $f(z)$ such that $f(z) \to \infty$ as $z \to z_0$ is called a pole of $f(z)$.*[7]

DEFINITION 2. *An isolated singular point z_0 of a function $f(z)$ such that $f(z)$ approaches no limit (finite or infinite) as $z \to z_0$ is called an essential singular point of $f(z)$.*

THEOREM 1.6. *The point z_0 is a pole of the function $f(z)$ if and only if it is a zero of the function*

$$\varphi(z) = \frac{1}{f(z)}.$$

Proof. If z_0 is a pole of $f(z)$, then $f(z) \to \infty$ as $z \to z_0$, and hence there is a deleted neighborhood $\mathcal{N}'(z_0)$ in which $|f(z)| > 1$. Obviously, $\varphi(z) = 1/f(z)$ is analytic on $\mathcal{N}'(z_0)$, and moreover $|\varphi(z)| < 1$ for all $z \in \mathcal{N}'(z_0)$. It follows from Theorem 1.5 that z_0 is a regular point of $\varphi(z)$, and that

$$\varphi(z_0) = \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

(why?). Therefore z_0 is a zero of $\varphi(z)$.

Conversely, let $\varphi(z)$ be single-valued and analytic in a neighborhood $\mathcal{N}(z_0)$, and suppose z_0 is a zero of $\varphi(z)$, where $\varphi(z) \not\equiv 0$. Then there is a number $\delta > 0$ such that the disk $|z - z_0| < \delta$ contains no zeros of $\varphi(z)$ other than z_0 itself (cf. p. I.371). Forming the function $f(z) = 1/\varphi(z)$, we see that $f(z)$ is analytic for $0 < |z - z_0| < \delta$ and approaches infinity as $z \to z_0$. In other words, z_0 is a pole of $f(z)$.

This result, which establishes a relation between poles and zeros, allows us to introduce the concept of the *order* of a pole. We say that the point z_0 is a *pole of order k* ($k \geqslant 1$) of the function $f(z)$ if z_0 is a zero of order k of the function $1/f(z)$. The pole is said to be *simple* if $k = 1$ and *multiple* if $k > 1$ (cf. p. I.161).

At a pole of order k, the Laurent expansion of $f(z)$ has the characteristic structure given by

THEOREM 1.7. *The point z_0 is a pole of order k of the function $f(z)$ if and only if the Laurent expansion of $f(z)$ at z_0 is of the form*

$$\begin{aligned} f(z) = a_{-k}(z - z_0)^{-k} + \cdots + a_{-1}(z - z_0)^{-1} \\ + a_0 + a_1(z - z_0) + \cdots, \end{aligned} \tag{1.23}$$

where $a_{-k} \neq 0$.

---

[7] Poles of elementary meromorphic functions have already been discussed in Chap. I.10, and the present considerations are the natural generalization of our previous results.

Proof. If $f(z)$ has a pole of order k at z_0, then $1/f(z)$ has a zero of order k at z_0, i.e.,

$$\frac{1}{f(z)} = A_k(z - z_0)^k + A_{k+1}(z - z_0)^{k+1} + \cdots,$$

where $A_k \neq 0$ (cf. p. I.371), and hence

$$f(z) = \frac{1}{(z - z_0)^k} \frac{1}{A_k + A_{k+1}(z - z_0) + \cdots}. \tag{1.24}$$

The power series

$$A_k + A_{k+1}(z - z_0) + \cdots$$

represents an analytic function which is nonvanishing in some neighborhood of z_0 (since $A_k \neq 0$). It follows that the function

$$\frac{1}{A_k + A_{k+1}(z - z_0) + \cdots}$$

is analytic in a neighborhood of z_0, and hence has an expansion of the form

$$\alpha_0 + \alpha_1(z - z_0) + \cdots,$$

where $\alpha_0 = 1/A_k \neq 0$. After substitution into (1.24), this gives

$$f(z) = \alpha_0(z - z_0)^{-k} + \alpha_1(z - z_0)^{-k+1} + \cdots \qquad (\alpha_0 \neq 0), \tag{1.25}$$

which is just the Laurent expansion of $f(z)$ at z_0 (see Corollary 2, p. 6). In fact, (1.25) coincides with (1.23) if we make the substitution $\alpha_n = a_{n-k}$ $(n = 0, 1, 2, \ldots)$.

Conversely, suppose $f(z)$ has an expansion of the form (1.23) at the point z_0, with $a_{-k} \neq 0$. Then

$$f(z) = \frac{a_{-k} + a_{-k+1}(z - z_0) + \cdots}{(z - z_0)^k},$$

which implies

$$\frac{1}{f(z)} = (z - z_0)^k \frac{1}{a_{-k} + a_{-k+1}(z - z_0) + \cdots}.$$

Replacing the function

$$\frac{1}{a_{-k} + a_{-k+1}(z - z_0) + \cdots}$$

by its Taylor series

$$\beta_0 + \beta_1(z - z_0) + \cdots,$$

we obtain

$$\frac{1}{f(z)} = (z - z_0)^k[\beta_0 + \beta_1(z - z_0) + \cdots]$$
$$= \beta_0(z - z_0)^k + \beta_1(z - z_0)^{k+1} + \cdots,$$

where $\beta_0 = 1/a_{-k} \neq 0$. It follows that z_0 is a zero of order k of the function $1/f(z)$, and hence a pole of order k of the original function $f(z)$. This completes the proof.

COROLLARY. *The point z_0 is an essential singular point of the function $f(z)$ if and only if the Laurent expansion of $f(z)$ at z_0 has infinitely many terms of the form $a_{-k}(z - z_0)^{-k}$, where $k > 0$, $a_{-k} \neq 0$.*

Proof. Obviously, z_0 is a regular point if and only if there are no negative powers of $z - z_0$ in the Laurent expansion of $f(z)$ at z_0. The case of finitely many negative powers is covered by Theorem 1.7. The only other possibility is that z_0 is an essential singular point.

Because of Theorem 1.7 and its corollary, we see that a decisive role is played by the negative powers of $z - z_0$ in the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n. \tag{1.26}$$

For this reason, the series of negative powers

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n = \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$$

is called the *principal part* of (1.26), while the series of nonnegative powers

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is called the *regular part* of (1.26).

Example 1. Consider the function

$$f(z) = e^{1/z^4}.$$

Clearly, the expansion

$$f(z) = 1 + \frac{1}{z^4} + \frac{1}{2!z^8} + \frac{1}{3!z^{12}} + \cdots \tag{1.27}$$

converges for all $z \neq 0$ and represents the Laurent expansion of $f(z)$ at the point $z = 0$. Since (1.27) contains infinitely many negative powers of z, the point $z = 0$ must be an essential singular point of $f(z)$. This is easily verified directly. In fact, $f(z) \to \infty$ if we approach $z = 0$ along the coordinate axes, while $f(z) \to 0$ if we approach $z = 0$ along the lines $y = \pm x$ bisecting the coordinate axes. Therefore $f(z)$ approaches no limit as $z \to 0$.

Example 2. Consider the Laurent series

$$\cdots + \frac{1}{z^n} + \frac{1}{z^{n-1}} + \cdots + \frac{1}{z} + \frac{z}{2^2} + \cdots + \frac{z^n}{2^{n+1}} + \cdots, \tag{1.28}$$

which contains infinitely many negative powers of z. Before we can assert that $z = 0$ is an essential singular point of the sum of (1.28), we must verify that (1.28) converges in a deleted neighborhood of $z = 0$. It is easy to see that (1.28) can be written in the form

$$\sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

Both series on the right are geometric series; the first converges to

$$\frac{\frac{1}{z}}{1 - \frac{1}{z}} = \frac{1}{z - 1}$$

for $|z| > 1$, while the second converges to

$$\frac{\frac{1}{2}}{1 - \frac{z}{2}} = \frac{1}{2 - z}$$

for $|z| < 2$. Therefore the original series (1.28) converges to the function

$$f(z) = \frac{1}{z - 1} + \frac{1}{2 - z} = \frac{-1}{z^2 - 3z + 2} \tag{1.29}$$

on the annulus D:$1 < |z| < 2$. Thus, since D is obviously not a deleted neighborhood of $z = 0$, it would be false to conclude that $z = 0$ is an essential singular point of $f(z)$. In fact, according to (1.29), $z = 0$ is a regular point of the function $f(z)$, whose only singular points are two simple poles, at the points $z = 1$ and $z = 2$.

The complicated behavior of a function at an essential singular point is shown by

THEOREM 1.8.[8] *If z_0 is an essential singular point of $f(z)$, then, given any complex number A (finite or infinite), there exists a sequence of points $\{z_n\}$ converging to z_0 such that*

$$\lim_{n \to \infty} f(z_n) = A. \tag{1.30}$$

Proof. We can assume that $A \neq \infty$, since the theorem is obvious if $A = \infty$ [$f(z)$ is unbounded in any deleted neighborhood of z_0].

[8] Variously known as the *Casorati-Weierstrass theorem* or *Sokhotski's theorem.*

Suppose the theorem is false, i.e., suppose that in an arbitrary deleted neighborhood $\mathcal{N}'(z_0)$ of the essential singular point z_0 it is impossible to find points at which the values of $f(z)$ are arbitrarily close to A. Then there must exist a number $\alpha > 0$ such that $|f(z) - A| > \alpha$ for all $z \in \mathcal{N}'(z_0)$. It follows that for all $z \in \mathcal{N}'(z_0)$ the function

$$\varphi(z) = \frac{1}{f(z) - A}$$

is analytic and satisfies the inequality

$$|\varphi(z)| = \frac{1}{|f(z) - A|} < \frac{1}{\alpha}.$$

Therefore, according to Theorem 1.5, z_0 is a regular point of $\varphi(z)$, and the value of $\varphi(z)$ at z_0 must equal

$$\lim_{z \to z_0} \frac{1}{f(z) - A}.$$

But $f(z)$ cannot be bounded in any deleted neighborhood of z_0, and therefore this limit can only equal zero, i.e., $\varphi(z)$ has a zero at z_0. Therefore $f(z) - A$ has a pole at z_0, and the same is true of $f(z)$ itself. This contradicts the assumption that z_0 is an essential singular point, thereby proving the theorem.

COROLLARY. *Let z_0 be an essential singular point of $f(z)$, and let E_δ be the set of values taken by $f(z)$ in the neighborhood $0 < |z - z_0| < \delta$. Then the closure of E_δ coincides with the extended complex plane.*

Example 1. The function

$$f(z) = \sin \frac{1}{z}$$

has an essential singular point at the origin. In fact, as $z \to 0$, $\sin(1/z)$ approaches no limit (finite or infinite), as can be seen at once by just considering real values of z. If $A = \infty$, the sequence $\{z_n\} = \{i/n\}$ satisfies the condition (1.30), since

$$f(z_n) = \sin \frac{1}{z_n} = \sin(-in) = -i \sinh n \to \infty \quad \text{as} \quad n \to \infty.$$

If $A \neq \infty$, we solve the equation

$$\sin \frac{1}{z} = A,$$

obtaining

$$\frac{1}{z} = \operatorname{Arc} \sin A = \frac{1}{i} \operatorname{Ln}(iA + \sqrt{1 - A^2}),$$

or

$$z = \frac{i}{\operatorname{Ln}(iA + \sqrt{1 - A^2})} = \frac{i}{\ln|iA + \sqrt{1 - A^2}| + 2k\pi i}.$$

Setting

$$z_n = \frac{i}{\ln|iA + \sqrt{1 - A^2}| + 2n\pi i} \qquad (n = 1, 2, \ldots),$$

we obtain a sequence $\{z_n\}$ which converges to zero and satisfies the condition (1.30), in fact the much stronger condition $f(z_n) = A$ $(n = 1, 2, \ldots)$.

Example 2. The function

$$f(z) = e^{1/z}$$

has an essential singular point at the origin, since, as already noted, $f(z)$ approaches no limit as $z \to 0$. If $A = \infty$, the sequence $\{z_n\} = \{1/n\}$ satisfies the condition (1.30), since

$$f(z_n) = e^n \to \infty \quad \text{as} \quad n \to \infty.$$

If $A = 0$, the sequence $\{z_n\} = \{-1/n\}$ is a sequence of the required type, since then

$$f(z_n) = e^{-n} \to 0 \quad \text{as} \quad n \to \infty.$$

Finally, assuming that $A \neq 0$, $A \neq \infty$, we solve the equation

$$e^{1/z} = A.$$

The result is

$$\frac{1}{z} = \operatorname{Ln} A,$$

which implies

$$z = \frac{1}{\operatorname{Ln} A} = \frac{1}{\ln|A| + 2k\pi i}.$$

Then, setting

$$z_n = \frac{1}{\ln|A| + 2n\pi i} \qquad (n = 1, 2, \ldots),$$

we obtain a sequence $\{z_n\}$ which converges to zero and satisfies the condition (1.30), in fact the much stronger condition $f(z_n) = A$ $(n = 1, 2, \ldots)$.

Remark. In each of the above examples, except for certain special values of A ($A = \infty$ in Example 1, $A = \infty$, $A = 0$ in Example 2), we can find a sequence $\{z_n\}$ which not only satisfies the requirement

$$\lim_{n \to \infty} f(z_n) = A$$

of Theorem 1.8, but also the much stronger condition of exact equality

$$f(z_n) = A \qquad (n = 1, 2, \ldots).$$

This is no accident, in view of the following result, known as *Picard's second* (or *great*) *theorem*, whose proof will not be given here: *If z_0 is an essential singular point of $f(z)$, then, given any complex number $A \neq \infty$, with the possible exception of a single value $A = A_0$, there exists a sequence of points $\{z_n\}$ converging to z_0 such that*

$$f(z_n) = A \qquad (n = 1, 2, \ldots).$$

In other words, the set E_δ figuring in the corollary to Theorem 1.8 coincides with the finite complex plane, with the possible exception of a single point A_0 (which is independent of δ). It is easy to see that this theorem implies Theorem 1.8. In Example 1, there is no exceptional value A_0, while in Example 2, $A_0 = 0$, since the function $e^{1/z}$ never vanishes.

3. Singular Points of $f(z) \pm g(z)$, $f(z)g(z)$ and $f(z)/g(z)$

Given a domain G, let $f(z)$ be a function with the property that every point of G is either a regular point or an isolated singular point of $f(z)$.[9] Suppose $g(z)$ is another function with the same property. We now use the results of the preceding section to investigate the singular points of the functions[10]

$$f(z) \pm g(z), \quad f(z)g(z), \quad f(z)/g(z). \tag{1.31}$$

Case 1. First suppose $f(z)$ and $g(z)$ are analytic on G, i.e., have no singular points at all in G. Then $f(z) \pm g(z)$ and $f(z)g(z)$ are obviously analytic on G. As for the function

$$F(z) = \frac{f(z)}{g(z)}, \tag{1.32}$$

it can have singular points only at the zeros of $g(z)$. Suppose z_0 is a zero of order $k \geqslant 0$ of $f(z)$ and a zero of order $l \geqslant 1$ of $g(z)$. [If z_0 is not a zero of $f(z)$, we set $k = 0$.] Then in a neighborhood of z_0 we have

$$F(z) = \frac{a_k(z - z_0)^k + \cdots}{b_l(z - z_0)^l + \cdots} = (z - z_0)^{k-l}\frac{a_k + \cdots}{b_l + \cdots},$$

where

$$a_k = \frac{f^{(k)}(z_0)}{k!} \neq 0, \qquad b_l = \frac{g^{(l)}(z_0)}{l!} \neq 0. \tag{1.33}$$

It follows that $F(z)$ has a pole of order $l - k$ at z_0 if $l > k$ and a regular point at z_0 if $l \leqslant k$, where the regular point is a zero of order $k - l$ if $l < k$.

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[9] From now on, we assume that all functions are single-valued, unless the contrary is explicitly stated.

[10] In the case of the function $f(z)/g(z)$, it is always assumed that $g(z) \not\equiv 0$.

Case 2. Suppose $f(z)$ and $g(z)$ are allowed to have poles in G. Then the situation is somewhat more complicated than in Case 1, since $f(z)$ and $g(z)$ now have Laurent expansions

$$f(z) = a_{-k}(z - z_0)^{-k} + \cdots + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + \cdots \quad (1.34)$$

and

$$g(z) = b_{-l}(z - z_0)^{-l} + \cdots + b_{-1}(z - z_0)^{-1} + b_0 + b_1(z - z_0) + \cdots \quad (1.35)$$

at the point z_0, instead of Taylor expansions. However, we might as well let k and l be arbitrary integers (positive, negative or zero). Then z_0 is a pole of order k of $f(z)$ if $k > 0$ and a regular point of $f(z)$ if $k \leqslant 0$, in particular, a zero of order k if $k < 0$, and similarly for $g(z)$. Clearly, the functions (1.31) can have no singular points other than poles in G.[11] For example, consider the difference $f(z) - g(z)$. Subtracting (1.35) from (1.34) term by term, we obtain the Laurent expansion of $f(z) - g(z)$ at z_0. The point z_0 will be a pole of $f(z) - g(z)$ if and only if

1. z_0 is a pole of at least one of the functions $f(z)$ and $g(z)$, i.e., $k \geqslant 1$ or $l \geqslant 1$;
2. The principal parts of $f(z)$ and $g(z)$ do not coincide.

However, if the principal parts do coincide, so that

$$k = l, \; a_{-k} = b_{-k}, \; \ldots, \; a_{-1} = b_{-1},$$

then

$$f(z) - g(z) = (a_0 - b_0) + (a_1 - b_1)(z - z_0) + \cdots,$$

and z_0 is a regular point of $f(z) - g(z)$.

Returning to the function (1.32), we note that it can now have singular points at the poles of $f(z)$ as well as at the zeros of $g(z)$. However, the conclusion of Case 1, i.e., the statement in italics, remains valid if we now permit k and l to be arbitrary integers.[12]

Case 3. Next we allow $f(z)$ but not $g(z)$ to have essential singular points as well as poles in G. If z_0 is an essential singular point of $f(z)$, then z_0 is an essential singular point of all the functions (1.31). In fact, if the principal part of $f(z)$ has infinitely many terms while that of $g(z)$ has only finitely many terms, the principal parts of the functions (1.31) will still have infinitely many terms, since none of the algebraic operations called for in (1.31) can eliminate more than a finite number of terms in the principal part of $f(z)$. There is another way of seeing this, which we illustrate by again considering the function (1.32). Suppose z_0 is an essential singular point of $f(z)$ and a

[11] This is almost obvious, since if the principal parts of $f(z)$ and $g(z)$ have only finitely many terms, the same must be true of the principal parts of the functions (1.31).

[12] Of course, (1.33) holds only if k and l are nonnegative.

regular point or pole of $g(z)$, but not an essential singular point of $F(z)$. Then since

$$f(z) = F(z)g(z),$$

it follows from Case 2 that z_0 cannot be an essential singular point of $f(z)$, contrary to hypothesis. This contradiction proves that z_0 is also an essential singular point of $F(z)$.

Case 4. Finally, we allow both $f(z)$ and $g(z)$ to have essential singular points as well as poles in G. First we note that if z_0 is an essential singular point of both $f(z)$ and $g(z)$, it need not be an essential singular point of the functions (1.31). For example, $f(z)$ and $g(z)$ may have the same principal parts and different regular parts at z_0, in which case $f(z) - g(z)$ has a regular point at z_0. Moreover, there is now an additional possibility: Let $f(z)$ have an essential singular point at z_0, and consider the function

$$\varphi(z) = \frac{1}{f(z)}.$$

Then there may or may not be a (deleted) neighborhood $\mathcal{N}'(z_0)$ containing no zeros of $f(z)$. For example, there is such a neighborhood if $f(z) = e^{1/z}$ but not if $f(z) = \sin(1/z)$. If there is such a neighborhood, then $\varphi(z)$ also has an essential singular point at z_0 (by virtually the same argument as in Case 3). However, if $f(z)$ has a zero in every neighborhood $\mathcal{N}'(z_0)$, then $\varphi(z)$ has a pole in every neighborhood $\mathcal{N}'(z_0)$, i.e., z_0 is no longer an isolated singular point of $\varphi(z)$, but rather a nonisolated singular point, namely, a *limit point of poles.*

4. Behavior at Infinity. The Poles of $g(z)\frac{d}{dz}\operatorname{Ln}[f(z) - A]$

Suppose $f(z)$ is analytic at every point of the unbounded domain $|z| > r$, except possibly at the point $z = \infty$ (see p. I.125). Then, making the preliminary transformation

$$\zeta = \frac{1}{z},$$

which carries the point $z = \infty$ into the point $\zeta = 0$, and every sequence $\{z_n\}$ converging to ∞ into a sequence $\{\zeta_n\} = \{1/z_n\}$ converging to 0, we reduce the study of $f(z)$ to that of the function

$$f^*(\zeta) = f(1/\zeta),$$

which is analytic in a deleted neighborhood of $z = 0$.[13]

---

[13] This device has been used repeatedly in Volume I, e.g., in Secs. I.24, I.43 and I.82. The last of these sections contains the definition of A-points (in particular, zeros) at infinity.

DEFINITION. *We say that $z = \infty$ is a regular point, a pole of order k or an essential singular point of $f(z)$, depending on whether $\zeta = 0$ is a regular point, a pole of order k or an essential singular point of $f^*(\zeta)$.*

It is an immediate consequence of the definition that if $z = \infty$ is a regular point of $f(z)$, then

$$f^*(\zeta) = a_0 + a_{-1}\zeta + \cdots + a_{-n}\zeta^n + \cdots,$$
$$f(z) = a_0 + a_{-1}z^{-1} + \cdots + a_{-n}z^{-n} + \cdots$$

Similarly, if $z = \infty$ is a pole of order k of $f(z)$,

$$f^*(\zeta) = a_k\zeta^{-k} + \cdots + a_1\zeta^{-1} + a_0 + a_{-1}\zeta + \cdots,$$
$$f(z) = a_kz^k + \cdots + a_1z + a_0 + a_{-1}z^{-1} + \cdots,$$

where $a_k \neq 0$, while if $z = \infty$ is an essential singular point of $f(z)$,

$$f^*(\zeta) = \sum_{n=-\infty}^{\infty} a_n\zeta^{-n},$$
$$f(z) = \sum_{n=-\infty}^{\infty} a_nz^n,$$

where infinitely many of the positive powers of z must be nonzero. In other words, the situation is the same as for finite points, except that the roles of positive and negative powers are now interchanged. Correspondingly, the principal part of a Laurent expansion at infinity consists of all terms with positive powers of z, and the regular part consists of all terms with non-positive powers of z.

As we know, it is not actually necessary to study the Laurent expansion of $f^*(\zeta)$ at $\zeta = 0$ in order to determine the behavior of $f^*(\zeta)$ at $\zeta = 0$. In fact, $f^*(\zeta)$ has a regular point at $\zeta = 0$ if $f^*(\zeta)$ is bounded in a deleted neighborhood $\mathcal{N}'(0)$, a pole at $\zeta = 0$ if

$$\lim_{\zeta \to 0} f^*(\zeta) = \infty,$$

and an essential singular point at $\zeta = 0$ if

$$\lim_{\zeta \to 0} f^*(\zeta)$$

fails to exist. Correspondingly, $f(z)$ has a regular point at $z = \infty$ if $f(z)$ is bounded in a deleted neighborhood $\mathcal{N}'(\infty)$, a pole at $z = \infty$ if

$$\lim_{z \to \infty} f(z) = \infty,$$

and an essential singular point at $z = \infty$ if

$$\lim_{z \to \infty} f(z)$$

fails to exist.

Example. Suppose $f(z)$ is an entire function, so that $f(z)$ is analytic at every finite point and has the power series expansion

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots \tag{1.36}$$

which converges for all finite z. The series (1.36) converges in a neighborhood $\mathcal{N}'(\infty)$ and can therefore be regarded as the Laurent expansion of $f(z)$ at $z = \infty$. There are now just three possibilities:

1. $f(z)$ is identically constant, i.e.,
$$f(z) \equiv a_0,$$
and has a regular point at ∞;
2. $f(z)$ reduces to a polynomial of degree k, i.e.,
$$f(z) = a_0 + a_1 z + \cdots + a_k z^k,$$
and has a pole of order k at ∞;
3. $f(z)$ has an essential singular point at ∞.

In the third case, we say that $f(z)$ is an entire *transcendental* function. In other words, an entire transcendental function approaches no limit (finite or infinite) as $z \to \infty$.[14]

The following result will be needed in Sec. 7:

THEOREM 1.9. *If z_0 is a regular point of $g(z)$ and an A-point or pole of $f(z)$, where $A \neq \infty$, then z_0 is a simple pole of the function*

$$g(z) \frac{d}{dz} \operatorname{Ln} [f(z) - A] = g(z) \frac{f'(z)}{f(z) - A}. \tag{1.37}$$

In fact, the Laurent expansion of (1.37) *has the form*

$$\frac{\alpha g(z_0)}{z - z_0} + \cdots$$

at every A-point of order α of $f(z)$, and the form

$$-\frac{\beta g(z_0)}{z - z_0} + \cdots$$

at every pole of order β of $f(z)$.

Proof. To keep things simple, we write only one leading term, using dots to indicate all higher-order terms. Suppose z_0 is an A-point of order α of $f(z)$. Then in a neighborhood of z_0 we have

$$\begin{aligned} g(z) &= g(z_0) + \cdots, \\ f(z) - A &= a(z - z_0)^\alpha + \cdots, \\ f'(z) &= a\alpha(z - z_0)^{\alpha - 1} + \cdots \end{aligned}$$

[14] An entire transcendental function was previously defined as an entire function, like e^z, which is not a polynomial (cf. p. I.135). We now see that it is not a coincidence that e^z approaches no limit as $z \to \infty$ (cf. p. I.140).

Denoting the function (1.37) by $F(z)$, we find that

$$\begin{aligned} F(z_0) &= [g(z_0) + \cdots] \frac{a\alpha(z - z_0)^{\alpha-1} + \cdots}{a(z - z_0)^{\alpha} + \cdots} \\ &= \frac{\alpha}{z - z_0} [g(z_0) + \cdots] \frac{1 + \cdots}{1 + \cdots} \\ &= \frac{\alpha}{z - z_0} [g(z_0) + \cdots] = \frac{\alpha g(z_0)}{z - z_0} + \cdots, \end{aligned}$$

as asserted.

On the other hand, suppose z_0 is a pole of order β of $f(z)$. Then z_0 is also a pole of order β of $f(z) - A$, and in a deleted neighborhood of z_0,

$$\begin{aligned} f(z) - A &= b(z - z_0)^{-\beta} + \cdots, \\ f'(z) &= -\beta b(z - z_0)^{-\beta-1} - \cdots, \end{aligned}$$

so that, in particular, z_0 is a pole of order $\beta + 1$ of $f'(z)$. It follows that

$$\begin{aligned} F(z_0) &= [g(z_0) + \cdots] \frac{-\beta b(z - z_0)^{-\beta-1} - \cdots}{b(z - z_0)^{-\beta} + \cdots} \\ &= \frac{-\beta}{z - z_0} [g(z_0) + \cdots] \frac{1 + \cdots}{1 + \cdots} \\ &= -\frac{\beta}{z - z_0} [g(z_0) + \cdots] = -\frac{\beta g(z_0)}{z - z_0} + \cdots, \end{aligned}$$

and the proof is complete.

*5. Dirichlet Series

We conclude this chapter, until now devoted to Laurent series, by studying another class of series related to power series. By a *Dirichlet series* we mean a series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \tag{1.38}$$

where the a_n are complex numbers and the λ_n are nonnegative real numbers (called the *exponents* of the series) satisfying the conditions

$$\lambda_n < \lambda_{n+1} \quad (n = 1, 2, \ldots), \quad \lim_{n \to \infty} \lambda_n = \infty. \tag{1.39}$$

In the special case where $\lambda_n = \ln n$, we have

$$e^{-\lambda_n z} = \frac{1}{n^z},$$

and (1.38) becomes

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z},$$

known as an *ordinary* (or *classical*) *Dirichlet series.*[15]

If we make the change of variable $w = e^{-z}$, the Dirichlet series (1.38) becomes

$$\sum_{n=1}^{\infty} a_n w^{\lambda_n},$$

which is a series expansion in arbitrary nonnegative powers of w. In general, the powers may be nonintegral, but in any event they must increase without limit as $n \to \infty$. From this point of view, a Dirichlet series can be regarded as a generalization of a power series to the case of arbitrary powers w^{λ_n} satisfying the conditions (1.39). Note that the case $\lambda_n = n$ corresponds to an ordinary power series.

Dirichlet series have many properties similar to those of power series but the region of convergence is now a half-plane instead of a disk, as shown by the next theorem (and the subsequent discussion):

THEOREM 1.10. *If the Dirichlet series* (1.38) *converges at the point* $z_0 = x_0 + iy_0$, *then it converges at every point of the half-plane* $\operatorname{Re} z = x > x_0$. *Moreover, the convergence is uniform on every closed domain* $\bar{G}_\theta$ *of the form*

$$|\arg (z - z_0)| \leqslant \theta < \frac{\pi}{2}$$

(*see Figure* 1.3).

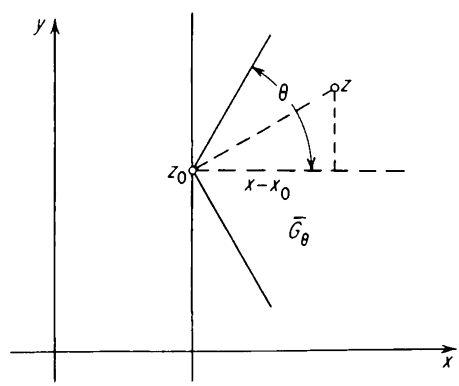

FIGURE 1.3

Proof. This theorem resembles Theorem I.16.2, except that it involves a half-plane instead of a disk, and in general, the convergence is not absolute. To prove the theorem, we estimate the quantity

$$\sum_{k=n+1}^{n+p} a_k e^{-\lambda_k z},$$

[15] Such series have many applications to number theory. See e.g., G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, third edition, Oxford University Press, London (1954).

using the fact that the series (1.38) converges for $z = z_0$. First we write $\alpha_n = 0$ and

$$\alpha_m = \sum_{k=n+1}^{m} a_k e^{-\lambda_k z_0} \qquad (m > n),$$

so that

$$a_k e^{-\lambda_k z_0} = \alpha_k - \alpha_{k-1} \qquad (k = n+1, n+2, \ldots).$$

Then we set

$$e^{-\lambda_k (z - z_0)} = b_k,$$

so that

$$\begin{aligned} \sum_{k=n+1}^{n+p} a_k e^{-\lambda_k z} &= \sum_{k=n+1}^{n+p} (\alpha_k - \alpha_{k-1}) b_k \\ &= \alpha_{n+p} b_{n+p} - \sum_{k=n+1}^{n+p-1} \alpha_k (b_{k+1} - b_k), \end{aligned} \tag{1.40}$$

where we have made an Abel transformation, as in equation (I.3.16). Given any $\varepsilon > 0$, there is an integer $N(\varepsilon)$ such that

$$|\alpha_m| = \left| \sum_{k=n+1}^{m} a_k e^{-\lambda_k z_0} \right| < \varepsilon \cos\theta \tag{1.41}$$

for all $n > N(\varepsilon)$ and $m > n$, since the series (1.38) converges at the point z_0. Moreover, we have

$$|b_{n+p}| = |e^{-\lambda_{n+p}(z - z_0)}| = e^{-\lambda_{n+p}(x - x_0)} \tag{1.42}$$

and

$$\begin{aligned} |b_{k+1} - b_k| &= \left| (z - z_0) \int_{\lambda_k}^{\lambda_{k+1}} e^{-t(z - z_0)}\, dt \right| \leqslant |z - z_0| \int_{\lambda_k}^{\lambda_{k+1}} e^{-t(x - x_0)}\, dt \\ &= \frac{|z - z_0|}{x - x_0} [e^{-\lambda_k (x - x_0)} - e^{-\lambda_{k+1}(x - x_0)}]. \end{aligned} \tag{1.43}$$

If $z \in \bar{G}_\theta$, then $x - x_0 > 0$ and

$$\frac{|z - z_0|}{x - x_0} \leqslant \frac{1}{\cos\theta} \tag{1.44}$$

(see Figure 1.3). Therefore, combining (1.40) and (1.44), we find that

$$\begin{aligned} \left| \sum_{k=n+1}^{n+p} a_k e^{-\lambda_k z} \right| &\leqslant \varepsilon \cos\theta\, e^{-\lambda_{n+p}(x - x_0)} \\ &\quad + \varepsilon \cos\theta \sum_{k=n+1}^{n+p-1} \frac{|z - z_0|}{x - x_0} [e^{-\lambda_k (x - x_0)} - e^{-\lambda_{k+1}(x - x_0)}] \\ &< \varepsilon\, e^{-\lambda_{n+1}(x - x_0)} < \varepsilon, \end{aligned}$$

provided that $n > N(\varepsilon)$. This proves both the uniform convergence of the Dirichlet series (1.38) on $\overline{G}_\theta$, and the ordinary convergence of (1.38) in the half-plane $x > x_0$.

Remark. A priori, a Dirichlet series can behave in any of the following ways:

1. The series may be convergent nowhere;
2. The series may be convergent everywhere;
3. The series may be convergent at some points but not at others.

In the third case, let $z_0 = x_0 + iy_0$ be a point at which the series converges, and let $z_1 = x_1 + iy_1$ be a point at which it diverges. It follows from Theorem 1.10 that $x_1 \leqslant x_0$ and that the series converges at all points of the half-plane $x > x_0$. Moreover, the series diverges at all points of the half-plane $x < x_1$ (otherwise it would converge at a point ζ with abscissa $\xi < x_1$, and hence, by Theorem 1.10, at the point z_1, contrary to hypothesis). It is possible that $x_1 = x_0$ (see Problem 1.17), and then the series converges if $x > x_0$ and diverges if $x < x_0$. If $x_1 < x_0$, let C be the greatest lower bound of the real parts of the points z for which the series converges; obviously, $x_1 \leqslant C \leqslant x_0$. Moreover, the series converges if $\operatorname{Re} z = x > C$ and diverges if $x < C$. To see this, we note that if $x > C$, then, by the definition of C, the series converges at some point ζ whose real part ξ satisfies the inequality $C \leqslant \xi < x$. But this implies that the series converges at z. Similarly, if $\operatorname{Re} z = x < C$, the assumption that the series converges at z contradicts the definition of C.

Therefore, the three ways in which a Dirichlet series can behave now take the following form:

1. The series may be convergent nowhere;
2. The series may be convergent everywhere;
3. There may exist a line $x = C$ such that the series is convergent at every point of the half-plane $x > C$ but not at any point of the half-plane $x < C$.

In the third case, the Dirichlet series is said to have *abscissa of convergence* C, or *half-plane of convergence* $x > C$. It is clear that the first two cases can be regarded as limiting cases of the third, corresponding to $C = +\infty$ and $C = -\infty$, respectively.

THEOREM 1.11. *The sum of the Dirichlet series* (1.38) *is an analytic function in the half-plane of convergence* $x > C$ $(C < \infty)$.

Proof. The series is uniformly convergent on every compact subset of the half-plane $x > C$ (why?), and every term $a_n e^{-\lambda_n z}$ is an entire function. The result now follows from Theorem I.15.8.

Remark 1. At first, one might expect that any function analytic in a half-plane $x > C$ can be represented by a Dirichlet series (1.38). However,

this conjecture, which is the converse of Theorem 1.11, is not true. For example, a necessary condition for a function to be the sum of the particularly simple Dirichlet series

$$\sum_{n=0}^{\infty} a_n e^{-nz} \tag{1.45}$$

is that it be periodic with period $2\pi i$, in addition to being analytic in a half-plane. However, this condition is not sufficient, i.e., there are functions of period $2\pi i$ which are analytic in a half-plane but cannot be represented as Dirichlet series of the form (1.45). In fact, given a function $f(z)$ of period $2\pi i$ which is analytic in a half-plane $x > C$, all we can say is that $f(z)$ has a representation of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{-nz} \qquad (\operatorname{Re} z > C). \tag{1.46}$$

To deduce (1.46), which generalizes (1.45) in the same way that a Laurent series generalizes a Taylor series, we carry out the mapping $\zeta = e^{-z}$. Under this mapping, $f(z)$ goes into a function $f^*(\zeta)$ which is single-valued and analytic on the disk $|\zeta| < e^{-C}$, except possibly at the center of the disk, It follows from Laurent's theorem (Theorem 1.3) that

$$f^*(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^n \qquad (0 < |\zeta| < e^{-C}).$$

Returning to the variable z, we find that (1.46) holds whenever $\operatorname{Re} z = x > C$. as asserted.

Remark 2. Just as the Fourier transform

$$\int_{-\infty}^{\infty} e^{izt}\varphi(t)\,dt$$

is the continuous analogue of the Fourier series

$$\sum_{n=-\infty}^{\infty} a_n e^{inz}$$

(written in complex form), so the Laplace transform

$$\int_{0}^{\infty} e^{-zt}\varphi(t)\,dt \tag{1.47}$$

is the continuous analogue of the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}. \tag{1.48}$$

In fact, as shown on p. I.422 (1.47) also defines a function analytic in a half-plane.

So far, we have said nothing about the absolute convergence of Dirichlet series. We now turn our attention to this subject:

THEOREM 1.12. *If the Dirichlet series* (1.48) *converges absolutely at the point* $z_0 = x_0 + iy_0$, *then it converges absolutely and uniformly in the closed half-plane* $\operatorname{Re} z = x \geqslant x_0$.

Proof. The theorem is an immediate consequence of the inequality

$$|a_n e^{-\lambda_n z}| = |a_n e^{-\lambda_n z_0}|\,|e^{-\lambda_n (z - z_0)}| = |a_n e^{-\lambda_n z_0}| e^{-\lambda_n (x - x_0)} \leqslant |a_n e^{-\lambda_n z_0}|$$

and the assumed convergence of the series

$$\sum_{n=1}^{\infty} |a_n e^{-\lambda_n z_0}|.$$

Remark. By essentially the same argument as given in the case of ordinary convergence, we see that a Dirichlet series must behave in one of the following three ways:

1′. The series may be absolutely convergent nowhere;
2′. The series may be absolutely convergent everywhere;
3′. There may exist a line $x = A$ such that the series is absolutely convergent at every point of the half-plane $x > A$, but not at any point of the half-plane $x < A$.

In the third case, the Dirichlet series is said to have *abscissa of absolute convergence* A, or *half-plane of absolute convergence* $x > A$. It is clear that the first two cases can be regarded as limiting cases of the third, corresponding to $A = +\infty$ and $A = -\infty$, respectively. Note that the series is either absolutely convergent at every point of the line $x = A$ or at no point of $x = A$.

THEOREM 1.13. *If the exponents* λ_n *satisfy the condition*[16]

$$L = \overline{\lim_{n \to \infty}} \frac{\ln n}{\lambda_n} < \infty, \tag{1.49}$$

as well as the conditions (1.39), *then*

$$0 \leqslant A - C \leqslant L, \tag{1.50}$$

where C *is the abscissa of convergence and* A *the abscissa of absolute convergence of the Dirichlet series* (1.48).

Proof. Since the point $z = C + \varepsilon$, where $\varepsilon > 0$ is arbitrary, belongs to the half-plane of convergence of the Dirichlet series, we have

$$\lim_{n \to \infty} a_n e^{-\lambda_n (C + \varepsilon)} = 0,$$

[16] Obviously, L is nonnegative. For an ordinary Dirichlet series, $\lambda_n = \ln n$ and $L = 1$, while in the case corresponding to a power series (see p. 27), $\lambda_n = n$ and $L = 0$.

and hence the sequence

$$\{a_n e^{-\lambda_n (C+\varepsilon)}\}$$

is bounded, i.e.,

$$|a_n e^{-\lambda_n (C+\varepsilon)}| \leqslant M \qquad (n = 1, 2, \ldots).$$

Therefore, for the point $z_0 = C + L + 3\varepsilon$ we have

$$\begin{aligned} |a_n e^{-\lambda_n z_0}| &= |a_n e^{-\lambda_n (C+L+3\varepsilon)}| \\ &= |a_n e^{-\lambda_n (C+\varepsilon)}| e^{-\lambda_n (L+2\varepsilon)} \leqslant M e^{-\lambda_n (L+2\varepsilon)}. \end{aligned} \tag{1.51}$$

But it follows from (1.49) that

$$\frac{\ln n}{\lambda_n} < L + \varepsilon$$

for sufficiently large n. In other words, there is an integer $N(\varepsilon) > 0$ such that $n > N(\varepsilon)$ implies

$$\ln n < \lambda_n (L + \varepsilon)$$

or

$$e^{-\lambda_n (L+2\varepsilon)} = [e^{-\lambda_n (L+\varepsilon)}]^{(L+2\varepsilon)/(L+\varepsilon)} < \frac{1}{n^{1+\delta}},$$

where

$$\delta = \frac{\varepsilon}{L + \varepsilon} > 0.$$

Then, according to (1.51),

$$|a_n e^{-\lambda_n z_0}| < \frac{M}{n^{1+\delta}}$$

if $n > N(\varepsilon)$, i.e., the Dirichlet series is absolutely convergent at the point $z_0 = C + L + 3\varepsilon$, which means that

$$C + L + 3\varepsilon \geqslant A.$$

Taking the limit of this inequality as $\varepsilon \to 0$, we obtain (1.50).[17]

Example. For the ordinary Dirichlet series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z}, \tag{1.52}$$

we have $L = 1$, so that

$$A - C \leqslant 1. \tag{1.53}$$

[17] The inequality $A - C \geqslant 0$ is obvious, since absolute convergence implies convergence.

Clearly, (1.52) diverges if $z = 0$. However, if $z = \delta > 0$, (1.52) is an alternating series whose terms decrease monotonically to zero in absolute value, and hence (1.52) converges, by Leibniz's theorem.[18] It follows that $C = 0$. On the other hand, since

$$\sum_{n=1}^{\infty} \frac{1}{n^{\delta}}$$

converges if $\delta > 1$ and diverges if $0 \leqslant \delta \leqslant 1$, the series (1.52) converges absolutely if $\delta > 1$, but not if $0 \leqslant \delta \leqslant 1$. Therefore $A = 1$, in keeping with (1.53). In other words, the series (1.52) is convergent, but not absolutely convergent, in the strip $0 < \operatorname{Re} z \leqslant 1$.

According to (1.50), $A = C$ when $L = 0$, e.g., when the Dirichlet series corresponds to a power series (cf. footnote 17). More generally, the abscissa of convergence and the abscissa of absolute convergence coincide for Dirichlet series whose exponents satisfy the condition

$$\overline{\lim_{n \to \infty}} \frac{\ln n}{\lambda_n} = 0, \tag{1.54}$$

as well as the conditions (1.39). We now prove the appropriate generalization of the Cauchy-Hadamard theorem (Theorem I.16.1) for a Dirichlet series of this kind:

THEOREM 1.14. *For a Dirichlet series* (1.48) *satisfying the conditions* (1.39) *and* (1.54), *the numbers A and C are given by the formula*

$$A = C = \overline{\lim_{n \to \infty}} \frac{\ln |a_n|}{\lambda_n}. \tag{1.55}$$

Proof. Suppose that

$$\overline{\lim_{n \to \infty}} \frac{\ln |a_n|}{\lambda_n} = \Lambda < \infty.$$

Let $z_0 = x_0 + iy_0$ be an arbitrary point in the half-plane $x > \Lambda$, and let

$$0 < \varepsilon < \tfrac{1}{3}(x_0 - \Lambda).$$

Then, by the definition of Λ,

$$\frac{\ln |a_n|}{\lambda_n} < \Lambda + \varepsilon \quad \text{or} \quad |a_n| < e^{\lambda_n(\Lambda + \varepsilon)}$$

if n exceeds a suitable positive integer $N_1(\varepsilon)$. Therefore

$$|a_n e^{-\lambda_n z_0}| = |a_n| e^{-\lambda_n x_0} < e^{-\lambda_n(x_0 - \Lambda - \varepsilon)} < e^{-2\lambda_n \varepsilon}$$

[18] See e.g., D. V. Widder, *Advanced Calculus*, second edition, Prentice-Hall, Inc., Englewood Cliffs, N. J. (1961), Theorem 8, p. 293.

if $n > N_1(\varepsilon)$. Moreover, (1.54) implies

$$\frac{\ln n}{\lambda_n} < \varepsilon,$$

i.e.,

$$n < e^{\lambda_n \varepsilon} \quad \text{or} \quad e^{-\lambda_n \varepsilon} < \frac{1}{n}$$

if n exceeds a suitable positive integer $N_2(\varepsilon)$, and hence

$$|a_n e^{-\lambda_n z_0}| < \frac{1}{n^2}$$

if $n > N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. This means that the Dirichlet series converges absolutely at z_0. Since z_0 is an arbitrary point of the half-plane $x > \Lambda$, it follows that

$$A = C \leqslant \Lambda. \tag{1.56}$$

Now let $z_1 = x_1 + iy_1$ be an arbitrary point of the half-plane $x < \Lambda$. Then, for any ε satisfying the inequality $0 < \varepsilon < \Lambda - x_1$, there is a sequence $\{n_k\}$ of positive integers, where $n_1 < \cdots < n_k < \cdots$, such that

$$\frac{\ln |a_{n_k}|}{\lambda_{n_k}} > \Lambda - \varepsilon > x_1$$

or

$$|a_{n_k}| > e^{\lambda_{n_k} x_1}.$$

It follows that

$$|a_{n_k} e^{-\lambda_{n_k} z_1}| = |a_{n_k}| e^{-\lambda_{n_k} x_1} > e^{\lambda_{n_k} x_1} e^{-\lambda_{n_k} x_1} = 1,$$

which means that the Dirichlet series (1.48) diverges at z_1 (since its general term cannot approach zero as $n \to \infty$). Therefore $\Lambda \leqslant A = C$, which together with (1.56) implies

$$A = C = \Lambda,$$

thereby proving the theorem.

PROBLEMS

1.1. Suppose the series (1.1) converges to $f(z)$ for all z such that $r < |z - z_0| < \infty$, and let $f^*(\zeta) = f(1/\zeta)$, $f^*(0) = A_0$. Prove that $f^*(\zeta)$ is analytic at $\zeta = 0$.

Hint. If $z_0 \neq 0$, consider the series

$$f^*(\zeta) = \sum_{n=0}^{\infty} A_n \left(\frac{\zeta}{1 - \zeta z_0}\right)^n,$$

where $|\zeta| < \min(r, |z_0|^{-1})$.

1.2. Prove that if the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

represents an even function, then

$$a_{2k+1} = 0 \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

while if the series represents an odd function, then

$$a_{2k} = 0 \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

1.3. Show that the Cauchy inequalities [cf. (I.16.43)] continue to hold for the coefficients a_n $(n = 0, \pm 1, \pm 2, \ldots)$ of a Laurent series.

1.4. Suppose $f(z)$ has a Laurent expansion at $z = \infty$, so that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (r < |z - z_0| < \infty).$$

Prove that $f(z)$ can also be represented in the form

$$f(z) = \sum_{n=-\infty}^{\infty} \tilde{a}_n z^n \qquad (\tilde{r} < |z| < \infty),$$

i.e., the circle figuring in the deleted neighborhood of infinity (see p. 10) can always be chosen with its center at the origin.

1.5. Expand each of the following functions $f(z)$ in a Laurent series on the indicated domain:

a) $\dfrac{1}{(z-a)(z-b)}$ $(0 < |a| < |b|)$ on the annulus $|a| < |z| < |b|$;

b) $\dfrac{z^2 - 2z + 5}{(z-2)(z^2+1)}$ on the annulus $1 < |z| < 2$;

c) $\sqrt{\dfrac{z}{(z-1)(z-2)}}$ $(\operatorname{Im} f(\tfrac{3}{2}) > 0)$ on the annulus $1 < |z| < 2$;

d) $e^{z+(1/z)}$ on the domain $0 < |z| < \infty$.

Ans.

b) $2\sum_{n=1}^{\infty} (-1)^n \dfrac{1}{z^{2n}} - \sum_{n=0}^{\infty} \dfrac{z^n}{2^{n+1}}$ for $1 < |z| < 2$;

d) $\sum_{n=0}^{\infty} c_n z^n + \sum_{n=1}^{\infty} c_{-n} z^{-n}$ for $0 < |z| < \infty$, where

$$c_n = c_{-n} = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!}.$$

1.6. Expand each of the following functions in a Laurent series at the indicated points, in each case finding suitable deleted neighborhoods in which the series converge:

a) $\dfrac{1}{z-2}$ at the points $z = 0$ and $z = \infty$;

b) $\dfrac{1}{(z-a)^k}$ $(a \neq 0$, integral $k > 0)$ at the points $z = 0$ and $z = \infty$;

c) $\dfrac{1}{(z^2+1)^2}$ at the points $z = i$ and $z = \infty$;

d) $\sqrt{(z-a)(z-b)}$ $(|a| \leqslant |b|)$ at the point $z = \infty$ (consider both branches);

e) $e^{1/(1-z)}$ at the points $z = 1$ and $z = \infty$;

f) $z^2 e^{1/z}$ at the points $z = 0$ and $z = \infty$.

Ans.

b) $\dfrac{(-1)^k}{a^k}\sum\limits_{n=0}^{\infty}\binom{n+k-1}{k-1}\left(\dfrac{z}{a}\right)^n$ for $|z| < |a|$, $\dfrac{1}{z^k}\sum\limits_{n=0}^{\infty}\binom{n+k-1}{k-1}\left(\dfrac{a}{z}\right)^n$ for $|z| > |a|$;

d) $\pm\left[z - \dfrac{1}{2}(a+b) + \sum\limits_{n=1}^{\infty}\dfrac{c_{-n}}{z^n}\right]$ for $|z| > |b|$, where

$$c_{-(n-1)} = (-1)^n\left[\binom{\frac{1}{2}}{n}b^n + \binom{\frac{1}{2}}{n-1}\binom{\frac{1}{2}}{1}b^{n-1}a + \binom{\frac{1}{2}}{n-2}\binom{\frac{1}{2}}{2}b^{n-2}a + \cdots + \binom{\frac{1}{2}}{n}a^n\right];$$

f) $\dfrac{1}{2} + z + z^2 + \sum\limits_{n=1}^{\infty}\dfrac{1}{(n+2)!z^n}$ for $0 < |z| < \infty$.

1.7. Find the Laurent expansion of the function

$$\ln\frac{z-a}{z-b}$$

at $z = \infty$.

Ans.

$$\sum_{n=1}^{\infty}\frac{b^n - a^n}{nz^n} \text{ for } |z| > \max(|a|, |b|).$$

1.8. Find the Laurent expansion of $\cot z$ on the annulus $\pi < |z| < 2\pi$.

Ans.

$$\cot z = \frac{3}{z} + 2\sum_{n=1}^{\infty}\frac{\pi^{2n}}{z^{2n+1}} + \sum_{n=1}^{\infty}\left[(-1)^n\frac{2^{2n}B_{2n}}{(2n)!} + \frac{2}{\pi^{2n}}\right]z^{2n-1},$$

where the B_{2n} denote the Bernoulli numbers introduced in Sec. I.90.

Comment. For the Laurent expansion of $\cot z$ on the annulus $0 < |z| < \pi$, see formula (I.18.32).

1.9. Find the singular points and investigate the behavior at infinity of the following functions:

a) $\dfrac{1}{z-z^3}$; b) $\dfrac{z^4}{1+z^4}$; c) $\dfrac{z^5}{(1-z)^2}$; d) $\dfrac{1}{z(z^2+4)^2}$; e) $\dfrac{e^z}{1+z^2}$;

f) $\dfrac{z^2+1}{e^z}$; g) ze^{-z}; h) $\dfrac{1}{e^z-1} - \dfrac{1}{z}$; i) $\dfrac{e^z}{z(1-e^{-z})}$;

j) $\dfrac{1-e^z}{1+e^z}$; k) $\tanh z$; l) e^{-1/z^2}; m) $ze^{1/z}$; n) $e^{z/(1-z)}$;

o) $e^{z-(1/z)}$; p) $\dfrac{e^{1/(z-1)}}{e^z-1}$.

Ans.

a) Simple poles at $0, \pm 1$; ∞ is a regular point, in fact a simple zero;

d) Simple pole at 0 and poles of order 2 at $\pm 2i$; ∞ is a regular point, in fact a zero of order 5;

g) ∞ is an essential singular point;
j) Simple poles at $(2k + 1)\pi i$, where $k = 0, \pm 1, \pm 2, \ldots$; ∞ is a limit point of poles;
m) 0 is an essential singular point; ∞ is a simple pole;
p) Simple poles at $2k\pi i$, where $k = 0, \pm 1, \pm 2, \ldots$; ∞ is a limit point of poles.

1.10. Find the singular points and investigate the behavior at infinity of the following functions:

a) $\dfrac{\cos z}{z^2}$; b) $\tan z$; c) $\tan^2 z$; d) $\dfrac{\cot z}{z^2}$; e) $\cot z - \dfrac{1}{z}$; f) $\cot z - \dfrac{2}{z}$;

g) $\dfrac{1}{\sin z - \sin a}$; h) $\dfrac{1}{\cos z + \cos a}$; i) $\sin \dfrac{1}{1 - z}$; j) $\cot \dfrac{1}{z}$;

k) $\cot \dfrac{1}{z} - \dfrac{1}{z}$; l) $\sin \dfrac{1}{z} + \dfrac{1}{z^2}$; m) $e^{-z} \cos \dfrac{1}{z}$; n) $e^{\cot(1/z)}$; o) $e^{\tan(1/z)}$;

p) $\sin\left(\dfrac{1}{\sin \dfrac{1}{z}}\right)$; q) $\sin\left(\dfrac{1}{\cos \dfrac{1}{z}}\right)$.

Ans.

b) Simple poles at $(2k + 1)\dfrac{\pi}{2}$, where $k = 0, \pm 1, \pm 2, \ldots$; ∞ is a limit point of poles;
e) Simple poles at $k\pi$, where $k = 0, \pm 1, \pm 2, \ldots$; ∞ is a limit point of poles;
h) Simple poles at $(2k + 1)\pi \pm a$, where $k = 0, \pm 1, \pm 2, \ldots$, if a is not an integral multiple of π, but otherwise poles of order 2 at $2k\pi$ if a is an odd multiple of π or at $(2k + 1)\pi$ if a is an even multiple of π; in every case, ∞ is a limit point of poles;
k) Simple poles at $1/k\pi$, where $k = \pm 1, \pm 2, \ldots$, and a limit point of poles at 0; ∞ is a simple pole;
n) Essential singular points at $1/k\pi$, where $k = \pm 1, \pm 2, \ldots$, and a limit point of essential singular points at 0; ∞ is an essential singular point;
q) Essential singular points at $2/(2k + 1)\pi$, where $k = 0, \pm 1, \pm 2, \ldots$, and a limit point of essential singular points at 0; ∞ is a regular point.

1.11. Prove that Theorem 1.8 remains valid if z_0 is a limit point of poles.

Comment. Accordingly, a limit point of poles is sometimes classified as an essential singular point.

1.12. Verify Picard's second theorem (p. 21) for each of the functions e^z and $\cos(1/z)$. What are the exceptional values, if any?

1.13. Prove that if the series

$$\sum_{n=1}^{\infty} z_n \quad \text{and} \quad \sum_{n=1}^{\infty} z'_n$$

are absolutely convergent, with sums s and s', respectively, then the series

$$\sum_{n=1}^{\infty} \left(\sum_{pq=n} z_p z'_q \right)$$

is absolutely convergent, with sum ss'. Here

$$\sum_{pq=n} z_p z'_q$$

denotes the sum of all distinct terms of the form $z_p z'_q$ whose indices have the product n.

1.14. Verify that the method of grouping terms in the preceding problem corresponds to the rule for formal multiplication of ordinary Dirichlet series. In other words, show that

$$\left(\sum_{n=1}^{\infty} \frac{a_n}{n^z}\right)\left(\sum_{n=1}^{\infty} \frac{b_n}{n^z}\right) = \sum_{n=1}^{\infty} \frac{c_n}{n^z},$$

where

$$c_n = \sum_{pq=n} a_p b_q.$$

Comment. In the same way, the method of grouping terms in Theorem I.3.7 corresponds to the rule for formal multiplication of power series.

1.15. Consider the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

discussed in Problem I.18.1. Prove that

$$\zeta^2(z) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^z},$$

where, as on p. I.431, $\tau(n)$ is the number of positive integers which divide n (including 1 and n).

1.16. Prove that the function $\zeta(z)$ of the preceding problem can be represented in the form

$$\zeta(z) = \frac{1}{\prod_{n=1}^{\infty} (1 - p_n^{-z})},$$

where p_n is the nth prime number greater than 1 ($p_1 = 2, p_2 = 3, p_3 = 5, \ldots$) and the infinite product is uniformly convergent on every compact subset of the half-plane D: $\operatorname{Re} z > 1$. Prove that $\zeta(z)$ has no zeros in D.

1.17. Give an example of a Dirichlet series which converges at a point z_0 and diverges at a point z_1 with the same abscissa (real part) as z_0.

Ans. $\sum_{n=1}^{\infty} \frac{1}{n} e^{-nz}$ converges if $z = \pi i$ but diverges if $z = 0$.

1.18. Give an example of a Dirichlet series for which

$$0 < \overline{\lim_{n \to \infty}} \frac{\ln n}{\lambda_n} < \infty$$

(cf. Theorem 1.13), but for which the abscissa of convergence and the abscissa of absolute convergence are the same.

1.19. Verify that the usual form of the Cauchy-Hadamard theorem (Theorem I.16.1) is a special case of Theorem 1.14.

1.20. Find the abscissa of convergence C and the abscissa of absolute convergence A of each of the following Dirichlet series:

a) $\sum_{n=0}^{\infty} e^{-n^3} e^{-zn^2}$; b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n} e^{-z \ln \ln n}$; c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} e^{-z \ln \ln n}$;

d) $\sum_{n=2}^{\infty} (-1)^n e^{-z \ln \ln n}$; e) $\sum_{n=1}^{\infty} (-1)^n e^{-z \ln n}$; f) $\sum_{n=1}^{\infty} \frac{1}{n^2} e^{-z \ln n}$;

g) $\sum_{n=0}^{\infty} e^{e^n} e^{-zn^2}$.

Ans. b) $C = -\infty$, $A = 1$; d) $C = 0$, $A = +\infty$; f) $C = A = -1$.

1.21. How does each of the following Dirichlet series behave on the boundary of its half-plane of convergence:

a) $\sum_{n=1}^{\infty} (-1)^n e^{-z \ln n}$; b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-zn}$; c) $\sum_{n=1}^{\infty} \frac{1}{n^3} e^{-z \ln n}$?

Ans. b) $C = A = 0$, and the series is absolutely convergent at every point of the boundary.

2

THE CALCULUS OF RESIDUES AND ITS APPLICATIONS

6. The Residue Theorem

Given a closed rectifiable Jordan curve L, suppose $f(z)$ is analytic on $\overline{I(L)}$ except for isolated singular points belonging to $I(L)$. Our aim is to evaluate the integral

$$\int_L f(z)\,dz.$$

First we note that $I(L)$ can contain no more than a finite number of singular points of $f(z)$. In fact, if $I(L)$ contained infinitely many singular points, these points would have a limit point $\zeta \in \overline{I(L)}$. But then ζ would be a nonisolated singular point (why?), contrary to hypothesis.

DEFINITION. *Suppose $f(z)$ is analytic in a deleted neighborhood $\mathcal{N}'(z_0)$ of an isolated singular point z_0. Then by the residue of $f(z)$ at z_0, denoted by*

$$\operatorname*{Res}_{z=z_0} f(z),$$

we mean the coefficient a_{-1} in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \qquad z \in \mathcal{N}'(z_0).$$

Remark. The notion of a residue was introduced by the French mathematician Cauchy (1789–1857), who arrived at this concept as early as the year 1814, while studying the difference between two integrals

$$\int_{L_1} f(z)\,dz, \qquad \int_{L_2} f(z)\,dz,$$

when the curves L_1 and L_2 have the same end points and there are poles of $f(z)$ in the region between L_1 and L_2. The term *residue* itself (French: *résidu*) seems to date back to 1826, and is also due to Cauchy. Subsequently, during the period 1826–1829, Cauchy applied the theory of residues to such problems as the evaluation of integrals, the expansion of functions in infinite series and products, the theory of equations, etc.

THEOREM 2.1 (*Residue theorem*). *Given a closed rectifiable Jordan curve L, if $f(z)$ is analytic on $\overline{I(L)}$, except for isolated singular points $z_1, \ldots, z_n$ belonging to $I(L)$, then*

$$\int_L f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Proof. Around each point z_k $(k = 1, \ldots, n)$, we draw a circle $\gamma_k: |z - z_k| = \rho_k$, choosing the radii ρ_k so small that

1. $I(L)$ contains every circle γ_k;
2. For every $k = 1, \ldots, n$, $E(\gamma_k)$ contains γ_j $(j \neq k)$.

Then it follows from Cauchy's integral theorem for a system of contours (Theorem I.13.4) that

$$\int_L f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz,$$

where all the integrals are taken in the same direction (as a rule, counterclockwise).

Now suppose the Laurent expansion of $f(z)$ at z_k is

$$f(z) = \sum_{m=-\infty}^{\infty} a_m^{(k)}(z - z_k)^m \qquad (k = 1, \ldots, n). \tag{2.1}$$

Then, integrating (2.1) term by term along γ_k (which is permissible because of the uniform convergence of the series on γ_k), we find that

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} \sum_{m=-\infty}^{\infty} a_m^{(k)}(z - z_k)^m\,dz = \sum_{m=-\infty}^{\infty} a_m^{(k)} \int_{\gamma_k} (z - z_k)^m\,dz$$
$$(k = 1, \ldots, n). \tag{2.2}$$

Since

$$\int_{\gamma_k} (z - z_k)^m \, dz = \int_0^{2\pi} (\rho_k e^{i\theta})^m \, d(\rho_k e^{i\theta})$$
$$= i\rho_k^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} \, d\theta = \begin{cases} 2\pi i & \text{if } m = -1, \\ 0 & \text{otherwise,} \end{cases}$$

(2.2) reduces to

$$\int_{\gamma_k} f(z) \, dz = 2\pi i a_{-1}^{(k)} \qquad (k = 1, \ldots, n). \tag{2.3}$$

Comparing (2.1) and (2.3), we have

$$\int_L f(z) \, dz = \sum_{k=1}^{m} \int_{\gamma_k} f(z) \, dz = 2\pi i \sum_{k=1}^{n} a_{-1}^{(k)} = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z),$$

and the theorem is proved.

Before we can apply Theorem 2.1, we must learn how to calculate residues. Suppose z_0 is a simple pole of $f(z)$, so that

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

in some deleted neighborhood of z_0. Then

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots,$$

and hence

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} [(z - z_0)f(z)]. \tag{2.4}$$

The calculation is particularly simple if $f(z)$ has the form

$$f(z) = \frac{\varphi(z)}{\psi(z)},$$

where $\varphi(z_0) \neq 0$ and $\psi(z)$ has a simple zero at $z = z_0$, i.e., $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$. Then z_0 is a simple pole of $f(z)$, and according to (2.4),

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{\varphi(z)}{\psi(z)} = \lim_{z \to z_0} \frac{(z - z_0)\varphi(z)}{\psi(z)} = \lim_{z \to z_0} \frac{\varphi(z)}{\dfrac{\psi(z) - \psi(z_0)}{z - z_0}} = \frac{\varphi(z_0)}{\psi'(z_0)}, \tag{2.5}$$

On the other hand, if z_0 is a pole of order $k > 1$ of $f(z)$, the Laurent expansion of $f(z)$ at z_0 is of the form

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots,$$

which implies

$$(z - z_0)^k f(z) = a_{-k} + \cdots + a_{-1}(z - z_0)^{k-1} + a_0(z - z_0)^k + a_1(z - z_0)^{k+1} + \cdots \tag{2.6}$$

Differentiating (2.6) $k-1$ times, we obtain

$$\frac{d^{k-1}}{dz^{k-1}}[(z-z_0)^k f(z)] = (k-1)!a_{-1} + \frac{k!}{1!}a_0(z-z_0) + \frac{(k+1)!}{2!}a_1(z-z_0)^2 + \cdots,$$

which implies

$$(k-1)!a_{-1} = \lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}[(z-z_0)^{k-1}f(z)]$$

or

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}[(z-z_0)^{k-1}f(z)]. \quad (2.7)$$

Example 1. Suppose $f(z)$ is a rational function

$$f(z) = \frac{P(z)}{Q(z)} = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n} \quad (a_m \neq 0,\ b_n \neq 0)$$

such that $n - m \geqslant 2$, with no poles on the real axis. To evaluate the integral

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

we choose the contour of integration L shown in Figure 2.1, where BCA is a semicircle of radius R with center at the origin. Let the radius R be so large that all the poles of $f(z)$ in the upper-half plane lie inside the contour L, i.e., belong to $I(L)$. Then, according to Theorem 2.1,

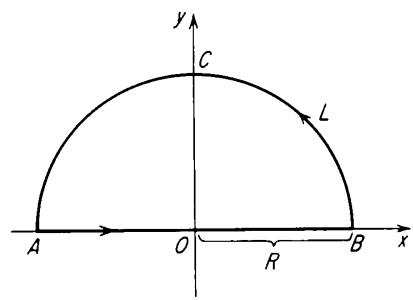

FIGURE 2.1

$$\int_{-R}^{R} f(x)\,dx + \int_{BCA} f(z)\,dz = 2\pi i \sum \operatorname{Res} f(z),$$

where the sum is over all poles of $f(z)$ lying in the upper half-plane. Since[1]

$$|f(z)| = \frac{|P(z)|}{|Q(z)|} = \frac{|a_0 + a_1 z + \cdots + a_m z^m|}{|b_0 + b_1 z + \cdots + b_n z^n|} = \left|\frac{a_m}{b_n}\right| \frac{1}{|z|^{n-m}} \frac{\left|1 + \cdots + \dfrac{a_0}{a_m z^m}\right|}{\left|1 + \cdots + \dfrac{b_0}{b_n z^n}\right|} \quad (2.8)$$

and $n - m \geqslant 2$, we have

$$|f(z)| < \frac{2|a_m|}{|b_n|R^2} = \frac{M}{R^2}$$

[1] The modification of (2.8) if $a_0 = 0$ or $b_0 = 0$ is obvious.

for sufficiently large values of $|z| = R$, where $M > 0$ is a constant. Therefore

$$\left|\int_{BCA} f(z)\,dz\right| < \frac{M}{R^2}\pi R = \frac{M\pi}{R} \to 0 \quad \text{as} \quad R \to \infty,$$

which implies

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(x).$$

In other words, the integral

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

where $f(z)$ is a rational function with no poles on the real axis and with a zero of order 2 or higher at infinity (this is equivalent to the condition $n - m \geqslant 2$), equals $2\pi i$ times the sum of the residues of $f(z)$ at poles lying in the upper half-plane.

For example, suppose

$$f(z) = \frac{z^{2p} - z^{2q}}{1 - z^{2r}},$$

where p, q and r are nonnegative integers such that $p < r$, $q < r$, while $p - q$ and r are relatively prime. Then the degree of the denominator is at least 2 higher than that of the numerator, and all the poles of $f(z)$ are given by the formula

$$z = e^{k\pi i/r} \qquad (k = 1, \ldots, r-1, r+1, \ldots, 2r-1) \tag{2.9}$$

[the points ± 1 are not poles of $f(z)$, since the numerator and denominator have the common factor $1 - z^2$].[2] Among the poles (2.9), those at the points

$$z = e^{k\pi i/r} \qquad (k = 1, 2, \ldots, r-1) \tag{2.10}$$

lie in the upper half-plane. All these poles are simple, and hence, according to (2.5),

$$\operatorname*{Res}_{z=e^{k\pi i/r}} f(z) = \frac{e^{2p(k\pi i/r)} - e^{2q(k\pi i/r)}}{-2re^{(2r-1)(k\pi i/r)}} = \frac{1}{2r}\left[e^{(2q+1)(k\pi i/r)} - e^{(2p+1)(k\pi i/r)}\right].$$

It follows that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x^{2p} - x^{2q}}{1 - x^{2r}}\,dx &= \frac{\pi i}{r}\sum_{k=1}^{r-1}\left[e^{(2q+1)(k\pi i/r)} - e^{(2p+1)(k\pi i/r)}\right] \\ &= \frac{\pi}{r}\left(i\frac{1 + e^{(2q+1)(\pi i/r)}}{1 - e^{(2q+1)(\pi i/r)}} - i\frac{1 + e^{(2p+1)(\pi i/r)}}{1 - e^{(2p+1)(\pi i/r)}}\right) \\ &= \frac{\pi}{r}\left(\cot\frac{2p+1}{2r}\pi - \cot\frac{2q+1}{2r}\pi\right).\end{aligned} \tag{2.11}$$

[2] If $p - q$ and r are not relatively prime, there are other points among (2.9) and (2.10) which are not poles of $f(z)$, but this does not affect the final result (2.12) [see Problem 2.6].

Noting that the integrand is even, we obtain the formula

$$\int_0^\infty \frac{x^{2p} - x^{2q}}{1 - x^{2r}}\,dx = \frac{\pi}{2r}\left(\cot\frac{2p+1}{2r}\pi - \cot\frac{2q+1}{2r}\pi\right). \tag{2.12}$$

If $r = 2n$ and $q = p + n$ $(p < n)$, then (2.12) reduces to

$$\int_0^\infty \frac{x^{2p}}{1 + x^{2n}}\,dx = \frac{\pi}{2\sin\dfrac{2p+1}{2}\pi}.$$

Example 2. Let $f(z)$ be analytic in the closed upper half-plane $\operatorname{Re} z \geqslant 0$, except at a finite number of poles, none of which lies on the real axis, and suppose

$$\max_{0 \leqslant \theta \leqslant \pi} |f(Re^{i\theta})| = M(R) \to 0 \qquad \text{as} \quad R \to \infty. \tag{2.13}$$

Then for any $\lambda > 0$ we have the formula

$$\int_0^\infty [f(x)e^{i\lambda x} + f(-x)e^{-i\lambda x}]\,dx = 2\pi i \sum \operatorname{Res}\{f(z)e^{i\lambda z}\}, \tag{2.14}$$

where the sum is over all the poles of $f(z)$ lying in the upper half-plane. To prove (2.14), let L be the same contour of integration as in Example 1 (see Figure 2.1), and choose the radius R so large that all the poles of $f(z)$ in the upper half-plane lie inside L. It follows from Theorem 2.1 that

$$\int_{-R}^{R} f(x)e^{i\lambda x}\,dx + \int_{BCA} f(z)\,dz = 2\pi i \sum \operatorname{Res}\{f(z)e^{i\lambda z}\}.$$

Denoting the second integral on the left by $J(R)$, we easily see that

$$\lim_{R \to \infty} J(R) = 0.$$

In fact,

$$\begin{aligned}
|J(R)| &= \left|\int_{BCA} f(z)e^{i\lambda z}\,dz\right| = \left|\int_0^\pi f(Re^{i\theta}) \exp(i\lambda R\cos\theta - \lambda R\sin\theta) iRe^{i\theta}\,d\theta\right| \\
&\leqslant \int_0^\pi |f(Re^{i\theta})| e^{-\lambda R\sin\theta} R\,d\theta \leqslant 2RM(R)\int_0^{\pi/2} e^{-\lambda R\sin\theta}\,d\theta \\
&\leqslant 2RM(R)\int_0^{\pi/2} e^{-2\lambda R\theta/\pi}\,d\theta = \frac{\pi M(R)}{\lambda}(1 - e^{-\lambda R}) \\
&< \frac{\pi M(R)}{\lambda} \to 0 \qquad \text{as} \quad R \to \infty,
\end{aligned}$$

where we have used (2.13) and the inequality

$$\sin\theta \geqslant \frac{2}{\pi}\theta \qquad \left(0 \leqslant \theta \leqslant \frac{\pi}{2}\right) \tag{2.15}$$

[cf. (I.13.20)]. Therefore

$$\begin{aligned}
\lim_{R \to \infty} \int_{-R}^{R} f(x)e^{i\lambda x}\,dx &= \lim_{R \to \infty} \int_0^R [f(x)e^{i\lambda x} + f(-x)e^{-i\lambda x}]\,dx \\
&= \int_0^\infty [f(x)e^{i\lambda x} + f(-x)e^{-i\lambda x}]\,dx = 2\pi i \sum \operatorname{Res}\{f(z)e^{i\lambda z}\},
\end{aligned}$$

which agrees with (2.14).

Formula (2.14) reduces to

$$\int_0^\infty f(x)\cos\lambda x\,dx = \pi i\sum \operatorname{Res}\{f(z)e^{i\lambda z}\}$$

if $f(z)$ is an even function, and to

$$\int_0^\infty f(x)\sin\lambda x\,dx = \pi\sum \operatorname{Res}\{f(z)e^{i\lambda z}\}$$

if $f(z)$ is an odd function. For example, we have

$$\int_0^\infty \frac{\cos\lambda x}{x^2+a^2}\,dx = \pi i \operatorname*{Res}_{z=ai}\frac{e^{i\lambda z}}{z^2+a^2} = \frac{\pi e^{-\lambda a}}{2a},$$

$$\int_0^\infty \frac{x\sin\lambda x}{x^2+a^2}\,dx = \pi \operatorname*{Res}_{z=ai}\frac{ze^{i\lambda z}}{z^2+a^2} = \frac{\pi e^{-\lambda a}}{2}.$$

7. The Argument Principle. The Theorems of Rouché and Hurwitz

Our next result is an immediate consequence of Theorems 1.9 and 2.1:

THEOREM 2.2. *Given a closed rectifiable Jordan curve L, suppose $g(z)$ is analytic on $\overline{I(L)}$, while $f(z)$ is analytic on $\overline{I(L)}$ except for poles in $I(L)$ at the points $b_1,\ldots,b_n$. Moreover, suppose $f(z)$ has A-points $a_1,\ldots,a_m$ in $I(L)$, but none on L itself. Then*

$$\frac{1}{2\pi i}\int_L g(z)\frac{f'(z)}{f(z)-A}\,dz = \sum_{k=1}^{m}\alpha_k g(a_k) - \sum_{k=1}^{n}\beta_k g(b_k), \qquad (2.16)$$

where α_k is the order of a_k and β_k the order of b_k.

The first term in the right-hand side of (2.16) is the sum of the values of $g(z)$ at the A-points of $f(z)$, where each value is repeated a number of times equal to the order of the A-point. If we assume that each A-point of $f(z)$ is counted a number of times equal to its order, then the sum

$$\sum_{k=1}^{m}\alpha_k g(a_k)$$

can be regarded as just the sum of the values of $g(z)$ at the A-points of $f(z)$. Similarly, the sum

$$\sum_{k=1}^{n}\beta_k g(b_k)$$

can be regarded as the sum of the values of $g(z)$ at the poles of $f(z)$, if each pole of $f(z)$ is counted a number of times equal to its order. With this convention (which will be in force from now on), we can paraphrase Theorem 2.2 as follows:

THEOREM 2.2'. *Given a closed rectifiable Jordan curve L, suppose $g(z)$ is analytic on $\overline{I(L)}$, while $f(z)$ is analytic on $\overline{I(L)}$ except for poles in $I(L)$ and has no A-points on L. Then the integral*

$$\frac{1}{2\pi i}\int_L g(z)\frac{f'(z)}{f(z)-A}\,dz \tag{2.17}$$

equals the sum of the values of $g(z)$ at the A-points of $f(z)$ minus the sum of the values of $g(z)$ at the poles of $f(z)$.

Example 1. If $g(z) = z$, then

$$\frac{1}{2\pi i}\int_L \frac{zf'(z)}{f(z)-A}\,dz = \sum_{k=1}^{m}\alpha_k a_k - \sum_{k=1}^{n}\beta_k b_k,$$

i.e., the integral (2.17) is just the sum of the A-points of $f(z)$ inside L minus the sum of the poles of $f(z)$ inside L.

Example 2. If $g(z) = 1$, then

$$\frac{1}{2\pi i}\int_L \frac{f'(z)}{f(z)-A}\,dz = \sum_{k=1}^{m}\alpha_k - \sum_{k=1}^{n}\beta_k, \tag{2.18}$$

where the quantity on the right equals the number of A-points of $f(z)$ inside L minus the number of poles of $f(z)$ inside L.

Now suppose $f(z)$ has N zeros and P poles inside L, where each zero and pole is counted a number of times equal to its order, as already described. (In particular, N or P may vanish.) Then, setting $A = 0$ in (2.18), we have

$$\frac{1}{2\pi i}\int_L \frac{d}{dz}\operatorname{Ln} f(z)\,dz = \frac{1}{2\pi i}\int_L \frac{f'(z)}{f(z)}\,dz = N - P. \tag{2.19}$$

The integral on the left is called the *logarithmic residue* of $f(z)$ relative to the contour L. In other words, *the number of zeros of $f(z)$ inside L minus the number of poles of $f(z)$ inside L equals the logarithmic residue of $f(z)$ relative to L.*

The logarithmic residue of $f(z)$ relative to L has a simple geometric interpretation. Choosing any point $z_0 \in L$ as the initial and final point of the path of integration, we make one circuit around L in the positive (i.e., counterclockwise) direction. Then $\operatorname{Ln} f(z)$ varies continuously, and in general returns to z_0 with a value different from its original value at z_0. In fact, since

$$\operatorname{Ln} f(z) = \ln|f(z)| + i \operatorname{Arg} f(z),$$

the change in $\operatorname{Ln} f(z)$ is entirely due to the change in $\operatorname{Arg} f(z)$. Letting Φ_0 denote the original value of $\operatorname{Arg} f(z_0)$ and Φ_1 its value after the circuit around L, we have

$$\frac{1}{2\pi i}\int_L \frac{d}{dz}\operatorname{Ln} f(z)\,dz = \frac{1}{2\pi i}[\ln|f(z_0)| + i\Phi_1] - [\ln|f(z_0)| + i\Phi_0] = \frac{\Phi_1 - \Phi_0}{2\pi}. \tag{2.20}$$

Comparing (2.19) and (2.20), we find that[3]

$$N - P = \frac{\Phi_1 - \Phi_0}{2\pi} = \frac{1}{2\pi}\Delta_L \operatorname{Arg} f(z), \tag{2.21}$$

In other words, the number of zeros of $f(z)$ inside L minus the number of poles of $f(z)$ inside L equals $1/2\pi$ times the change in $\operatorname{Arg} f(z)$ when the contour L is traversed once in the positive direction.

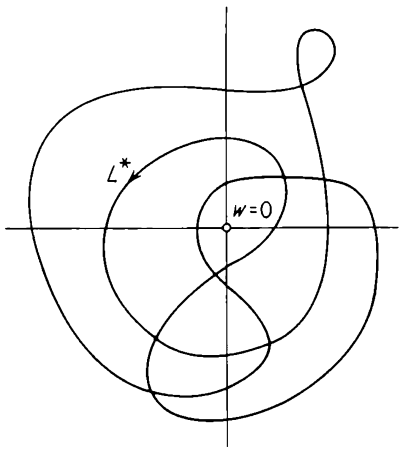

FIGURE 2.2

There is still another way of interpreting this result: As a variable point z describes the closed curve L once in the positive direction, the image point $w = f(z)$ describes a closed curve $L^* = f(L)$ in the w-plane, making some number ν of complete circuits around the origin $w = 0$, where every circuit made in the positive direction is counted as $+1$ and every circuit made in the negative direction is counted as -1. For example, $\nu = 1$ in Figure 2.2. It is easy to see that

$$\Delta_L \operatorname{Arg} f(z) = 2\pi\nu,$$

and therefore the number of zeros of $f(z)$ inside L minus the number of poles of $f(z)$ inside L equals the number of circuits around the origin made by the point $w = f(z)$ as the point z traverses the curve L once in the positive direction.

Making the obvious generalization to the case where $A \neq 0$ in (2.18), we summarize these results in the form of

THEOREM 2.3 (*Argument principle*). *Given a closed rectifiable Jordan curve L, suppose $f(z)$ is analytic on $\overline{I(L)}$ except for poles in $I(L)$ and has no A-points on L. Then the number of A-points of $f(z)$ inside L minus the number of poles of $f(z)$ inside L equals the number of circuits around the point $w = A$ made by the point $w = f(z)$ as the point z traverses the curve L once in the positive direction.*

We now turn to some implications of the argument principle:

THEOREM 2.4 (*Rouché's theorem*). *Given a closed rectifiable Jordan curve L, suppose $f(z)$ and $g(z)$ are analytic on $\overline{I(L)}$, and suppose that*

$$|f(z)| > |g(z)| \tag{2.22}$$

[3] By $\Delta_L h(z)$ is meant the change in $h(z)$ as z traverses the curve L. This quantity is sometimes denoted by $\operatorname{Var}_{z \in L} h(z)$ [cf. p. I.314].

at every point of L. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside L.

Proof. Since $f(z)$ cannot vanish on L, because of (2.22), we have

$$\begin{aligned}\Delta_L \operatorname{Arg}[f(z) + g(z)] &= \Delta_L\left\{\operatorname{Arg} f(z)\left[1 + \frac{g(z)}{f(z)}\right]\right\} \\ &= \Delta_L \operatorname{Arg} f(z) + \Delta_L \operatorname{Arg}\left[1 + \frac{g(z)}{f(z)}\right].\end{aligned} \tag{2.23}$$

Moreover, since

$$\left|\frac{g(z)}{f(z)}\right| < 1$$

for all $z \in L$, the variable point

$$w = 1 + \frac{g(z)}{f(z)}$$

stays in the disk $|w - 1| < 1$ as z describes the curve L. Therefore w cannot wind around the origin, which means that

$$\Delta_L \operatorname{Arg}\left[1 + \frac{g(z)}{f(z)}\right] = 0. \tag{2.24}$$

Combining (2.23) and (2.24), we find that

$$\Delta_L \operatorname{Arg}[f(z) + g(z)] = \Delta_L \operatorname{Arg} f(z),$$

and the theorem is now an immediate consequence of the argument principle.

THEOREM 2.5 (*Hurwitz's theorem*). *Given a closed rectifiable Jordan curve L, suppose $\{f_n(z)\}$ is a sequence of analytic functions on $\overline{I(L)}$ which converges uniformly on $\overline{I(L)}$ to a function $f(z) \not\equiv 0$. If L passes through no zeros of $f(z)$, then there is an integer $N(L) > 0$ such that every function $f_n(z)$ with $n > N(L)$ has the same number of zeros inside L as the function $f(z)$.*

Proof. Let μ be the minimum of $|f(z)|$ on $|L$; by hypothesis, $\mu > 0$. Therefore, since $\{f_n(z)\}$ is uniformly convergent on L, we can find an integer $N(L) > 0$ such that

$$|f_n(z) - f(z)| < \mu \leqslant |f(z)|$$

for all $n > N(L)$ and $z \in L$. But then, according to Rouché's theorem, the functions $f(z)$ and

$$f(z) + [f_n(z) - f(z)] = f_n(z), \qquad n > N(L)$$

have the same number of zeros inside L, as asserted.

Example 1. *Find the number of roots of the equation*

$$z^8 - 4z^5 + z^2 - 1 = 0 \tag{2.25}$$

of absolute value less than 1.

Writing the left-hand side of (2.25) in the form $f(z) + g(z)$, where

$$f(z) = -4z^5, \quad g(z) = z^8 + z^2 - 1,$$

we see that $|f(z)| > |g(z)|$ on the circle $|z| = 1$, since

$$|f(z)| = |4z^5| = 4, \quad |g(z)| = |z^8 + z^2 - 1| \leqslant |z^8| + |z^2| + 1 = 3$$

if $|z| = 1$. Therefore, according to Rouché's theorem, the function

$$f(z) + g(z) = z^8 - 4z^5 + z^2 - 1$$

has the same number of zeros inside the circle $|z| = 1$ as the function $f(z) = -4z^5$. But $f(z)$ obviously has five zeros inside $|z| = 1$, in fact, a zero of order five at the origin (recall our way of counting zeros and poles). In other words, equation (2.25) has five roots inside the unit circle, i.e., five roots of absolute value less than 1.

Example 2. *Prove that the equation*

$$a_0 + a_1 \cos\theta + \cdots + a_n \cos n\theta = 0, \tag{2.26}$$

where

$$0 < a_0 < a_1 < \cdots < a_n,$$

has $2n$ *distinct real roots in the interval* $0 < \theta < 2\pi$, *and no imaginary roots at all.*

We begin by showing that all the zeros of the polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

lie inside the unit circle $|z| = 1$. If z is not positive, then

$$\begin{aligned} |(z-1)p(z)| &= |a_n z^{n+1} - [a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n]| \\ &\geqslant \big|\, |a_n z^{n+1}| - |a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n| \,\big| \\ &> |\, |a_n z^{n+1}| - [a_0 + (a_1 - a_0)|z| + \cdots + (a_n - a_{n-1})|z|^n]|. \end{aligned} \tag{2.27}$$

In the last step, we have used the fact that the strict inequality

$$\begin{aligned} |a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n| \\ < a_0 + (a_1 - a_0)|z| + \cdots + (a_n - a_{n-1})|z|^n \end{aligned}$$

holds, since the vectors $a_0, (a_1 - a_0)z, \ldots, (a_n - a_{n-1})z^n$ cannot all have the same direction [the numbers $a_0, a_1 - a_0, \ldots, a_n - a_{n-1}$ (but not z) are all positive]. If, in addition, $|z| \geqslant 1$, then

$$\begin{aligned} a_0 + (a_1 - a_0)|z| + \cdots + (a_n - a_{n-1})|z|^n \\ \leqslant a_0|z|^{n+1} + (a_1 - a_0)|z|^{n+1} + \cdots + (a_n - a_{n-1})|z|^{n+1} \\ = [a_0 + (a_1 - a_0) + \cdots + (a_n - a_{n-1})]|z|^{n+1} = a_n|z|^{n+1}, \end{aligned}$$

and in this case, (2.27) becomes

$$|(z-1)p(z)| > a_n|z|^{n+1} - a_n|z|^{n+1} = 0,$$

i.e.,

$$(z-1)p(z) \neq 0.$$

In other words, $p(z) \neq 0$ if $|z| \geqslant 1$ and z is not positive. But $p(z)$ obviously has no positive zeros. Therefore all the zeros of $p(z)$ must lie *inside* the unit circle $|z| = 1$.

Now suppose the point z describes the circle $|z| = 1$ once in the positive direction. Then, according to the argument principle, the image point $w = p(z)$ must wind around the origin $w = 0$ as many times as there are zeros of $p(z)$, i.e., n times. Since each time w winds around the origin it must intersect the imaginary axis at least twice (once in the upper and once in the lower half-plane), there are at least $2n$ such intersections. Each intersection corresponds to a definite position of the point z on the circle $|z| = 1$, and hence to a definite value of $\theta = \arg z$, which varies from 0 to 2π as z traverses the circle $|z| = 1$. Thus there are at least $2n$ distinct values of θ in the interval $(0, 2\pi)$ for which the point $w = p(z) = p(e^{i\theta})$ lies on the imaginary axis, and for each such value of θ,

$$\begin{aligned}\operatorname{Re} p(e^{i\theta}) &= \operatorname{Re}(a_0 + a_1 e^{i\theta} + \cdots + a_n e^{in\theta}) \\ &= \operatorname{Re}[a_0 + a_1(\cos\theta + i\sin\theta) + \cdots + a_n(\cos n\theta + i\sin n\theta)] \\ &= a_0 + a_1\cos\theta + \cdots + a_n\cos n\theta = 0.\end{aligned}$$

Therefore there are at least $2n$ roots of the equation (2.26) in the interval $(0, 2\pi)$.

To show that there are precisely $2n$ roots of (2.26) in the interval $(0, 2\pi)$, we set $\zeta = e^{i\theta}$, obtaining

$$\cos k\theta = \frac{e^{ik\theta} + e^{-ik\theta}}{2} = \frac{\zeta^k + \zeta^{-k}}{2}$$

and

$$a_0 + a_1\cos\theta + \cdots + a_n\cos n\theta = \tfrac{1}{2}\zeta^{-n}q(\zeta),$$

where

$$q(\zeta) = a_n + a_{n-1}\zeta + \cdots + a_1\zeta^{n-1} + 2a_0\zeta^n + a_1\zeta^{n+1} + \cdots + a_n\zeta^{2n}.$$

The polynomial $q(\zeta)$ has precisely $2n$ zeros $\zeta_1, \zeta_2, \ldots, \zeta_{2n}$, and any possible root θ_0 of (2.26) must satisfy the relation

$$e^{i\theta_0} = \zeta_k \tag{2.28}$$

for some $k = 1, 2, \ldots, 2n$. In particular, there can be no more than $2n$ real roots of (2.26) in the interval $(0, 2\pi)$ [or in any interval of length 2π]. But the existence of $2n$ roots of (2.26) in $(0, 2\pi)$ has just been established, and hence the number of such roots is precisely $2n$. Since these $2n$ roots are distinct, so are the numbers $\zeta_1, \zeta_2, \ldots, \zeta_{2n}$, which obviously all have absolute value 1.

This last fact, together with (2.28), excludes the possibility of (2.26) having any imaginary roots.

Example 3. Let $\varphi(t) > 0$ be a strictly increasing continuous function in the interval [0, 1], and consider the integral

$$f(z) = \int_0^1 \varphi(t) \cos zt \, dt.$$

Then, just as in the proof of Theorem I.17.19, the sequence of approximating sums

$$f_n(z) = \sum_{k=1}^{n} \frac{1}{n} \varphi\left(\frac{k}{n}\right) \cos \frac{kz}{n}$$

converges uniformly to $f(z)$ on every compact set F.[4] Therefore $f(z)$ is an entire function, and moreover $f(z) \not\equiv 0$, since, for example,

$$f(0) = \int_0^1 \varphi(t) \, dt > 0.$$

Applying Hurwitz's theorem to the sequence $\{f_n(z)\}$ and the function $f(z)$, we see that if z_0 is any zero of $f(z)$, then in any fixed neighborhood $\mathcal{N}(z_0)$, all the functions $f_n(z)$ starting from a certain value of n must have the same number of zeros as $f(z)$, i.e., at least one zero. But the functions $f_n(z)$ are trigonometric polynomials satisfying the conditions of Example 2, where in the present case $\theta = z/n$ and

$$0 < a_k = \frac{1}{n} \varphi\left(\frac{k}{n}\right) < a_{k+1} = \frac{1}{n} \varphi\left(\frac{k+1}{n}\right).$$

Therefore the functions $f_n(z)$ can have no imaginary zeros, and hence the same is true of $f(z)$, i.e., all the zeros of $f(z)$ are real.

8. Residues at Infinity

Let $f(z)$ be analytic in a deleted neighborhood of the point at infinity, so that $f(z)$ has a Laurent expansion of the form

$$f(z) = \cdots + \frac{a_{-m}}{z^m} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots + a_n z^n + \cdots \qquad (R < |z| < \infty).$$

Suppose we integrate $f(z)$ along the circle $\gamma_\rho: |z| = \rho$, where $\rho > R$, choosing the following convention for the positive direction of traversing γ_ρ: An

[4] For a more elementary proof of this fact, note that $g(z, t) = \varphi(t) \cos zt$ is uniformly continuous on the (compact) set of all ordered pairs (z, t), where $z \in F$, $t \in [0, 1]$. In other words, there is a positive function $\omega(\varepsilon)$ such that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $|g(z, t) - g(z, t')| < \omega(\varepsilon)$ for all $z \in F$ and $t, t' \in [0, 1]$, provided that $|t - t'| < \varepsilon$. But then $|f_n(z) - f(z)| < \omega(1/n) \to 0$ as $n \to \infty$.

observer moving along γ_ρ in the positive direction finds the *exterior* of γ_ρ, and hence the point at infinity, on his left. In other words, the positive direction of traversing a curve surrounding the point at infinity is the *clockwise direction.*[5] It follows that

$$\int_{\gamma_\rho} f(z)\,dz = \int_{\gamma_\rho} \frac{a_{-1}}{z}\,dz = ia_{-1}\int_0^{-2\pi} d\theta = -2\pi i a_{-1}.$$

This suggests defining the quantity $-a_{-1}$ as the *residue of $f(z)$ at infinity*, since then

$$\int_{\gamma_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z),$$

in keeping with the residue theorem.

Remark. It should be noted that the residue of $f(z)$ at ∞ is determined by the coefficient of one of the terms of the *regular part* of the Laurent expansion of $f(z)$ at ∞, whereas the residue of $f(z)$ at a finite point z_0 is determined by the coefficient of one of the terms of the *principal part* of the Laurent expansion of $f(z)$ at z_0. [Recall that in defining regular and principal parts at ∞, as opposed to z_0, the roles of positive and negative powers of z are interchanged.] It follows that the residue of $f(z)$ at ∞ can be nonzero even when ∞ is a regular point of $f(z)$, whereas the residue at a finite regular point is always zero. For example, ∞ is a regular point of the function

$$f(z) = \frac{1}{z},$$

in fact, a simple zero, but

$$\operatorname*{Res}_{z=\infty} f(z) = -1 \neq 0.$$

THEOREM 2.6. *If $f(z)$ is analytic except at isolated singular points, then the sum of all the residues of $f(z)$ equals zero.*

Proof. The function $f(z)$ can only have a finite number of singular points, since otherwise the singular points of $f(z)$ would have a limit point ζ (possibly at infinity), and then ζ would be a nonisolated singular point of $f(z)$, contrary to hypothesis. Thus there is a circle $\gamma_\rho: |z| = \rho$ whose radius ρ is so large that all the finite singular points $z_1, \ldots, z_n$ of $f(z)$ lie inside γ_ρ. Then, according to the residue theorem,

$$\int_{\gamma_\rho} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z), \tag{2.29}$$

[5] Previously, in dealing with a contour L surrounding one or more finite points, we chose the *counterclockwise* direction as positive, since then an observer moving along L finds the *interior* of L, and hence the given finite points, on his left.

where γ_ρ is traversed in the usual positive direction for finite points, i.e., in the counterclockwise direction. But relative to the point at infinity, this is the negative direction of traversing γ_ρ, and hence the same integral equals

$$\int_{\gamma_\rho} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z). \tag{2.30}$$

Subtracting (2.30) from (2.29), we find that

$$2\pi i\Big[\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z)\Big] = 0,$$

or

$$\operatorname*{Res}_{z=z_1} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0,$$

as asserted.

Example. Theorem 2.6 applies to any rational function, since a rational function has only isolated singular points, namely, its poles.

9. Cauchy's Theorem on Partial Fraction Expansions

We now apply the residue theorem to the important problem of expanding functions in partial fractions. We begin by proving the following

LEMMA. *Let $f(z)$ be an analytic function whose only singular points in the finite plane are poles, and let L be any closed rectifiable Jordan curve such that*

1. *The origin lies inside L;*
2. *L passes through no poles of $f(z)$.*

Let $b_0 = 0, b_1, \ldots, b_n$ be the poles of $f(z)$ inside L, with principal parts[6]

$$G_k(z) = \frac{a_{-\beta_k}^{(k)}}{(z-b_k)^{\beta_k}} + \cdots + \frac{a_{-1}^{(k)}}{z-b_k} \qquad (k = 0, 1, \ldots, n),$$

where $G_0(z) \equiv 0$ if $b_0 = 0$ is a regular point of $f(z)$. Then

$$f(z) = \sum_{k=0}^{n} G_k(z) + \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta, \tag{2.31}$$

where z is an arbitrary regular point of $f(z)$ inside L.

---

[6] By the "principal part of a pole b of $f(z)$" we mean the "principal part of the Laurent expansion of $f(z)$ at the pole $z = b$." It is convenient to treat the pole of $f(z)$ at the origin (if any) separately, by reserving the symbol $b_0 = 0$ for this pole and $G_0(z)$ for its principal part. Thus, of all the principal parts $G_k(z)$, $k = 0, 1, \ldots, n$, only $G_0(z)$ can vanish identically, and this happens only when $f(z)$ has no pole at the origin.

Proof. The only poles inside L of the function

$$\varphi(\zeta) = \frac{f(\zeta)}{\zeta - z} \tag{2.32}$$

are at the points $b_0, b_1, \ldots, b_n$ and at the point $\zeta = z$. In a deleted neighborhood of the point b_k $(k = 0, 1, \ldots, n)$, we have

$$f(\zeta) = \frac{a_{-\beta_k}^{(k)}}{(\zeta - b_k)^{\beta_k}} + \cdots + \frac{a_{-1}^{(k)}}{\zeta - b_k} + a_0^{(k)} + a_1^{(k)}(\zeta - b_k) + \cdots, \tag{2.33}$$

while

$$\begin{aligned} \frac{1}{\zeta - z} &= -\frac{1}{(z - b_k) - (\zeta - b_k)} = -\frac{1}{z - b_k}\frac{1}{1 - \dfrac{\zeta - b_k}{z - b_k}} \\ &= -\frac{1}{z - b_k} - \frac{\zeta - b_k}{(z - b_k)^2} - \cdots - \frac{(\zeta - b_k)^{\beta_k - 1}}{(z - b_k)^{\beta_k}} - \cdots \end{aligned} \tag{2.34}$$

if $|\zeta - b_k| < |z - b_k|$. Multiplying (2.32) by (2.33), we find that the term proportional to $(\zeta - b_k)^{-1}$ in the Laurent expansion of the function (2.32) at the point $\zeta = b_k$ equals

$$-\left[\frac{a_{-\beta_k}^{(k)}}{(z - b_k)^{\beta_k}} + \cdots + \frac{a_{-1}^{(k)}}{z - b_k}\right]\frac{1}{\zeta - b_k} = -\frac{G_k(z)}{\zeta - b_k}.$$

In other words,

$$\operatorname*{Res}_{\zeta = b_k} \varphi(\zeta) = -G_k(z) \qquad (k = 0, 1, \ldots, n).$$

Since obviously

$$\operatorname*{Res}_{\zeta = z} \varphi(\zeta) = f(z),$$

it follows from the residue theorem that

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_L \varphi(\zeta)\,d\zeta = f(z) - \sum_{k=0}^{n} G_k(z),$$

which is equivalent to (2.31).

Remark. Another proof of the lemma starts with the observation that

$$\sum_{k=0}^{n} G_k(\zeta)$$

is a rational function which vanishes at infinity and has all its poles inside L. The same is true of the function

$$\frac{1}{\zeta - z}\sum_{k=0}^{n} G_k(\zeta), \qquad z \in I(L), \tag{2.35}$$

except that now infinity is known to be a zero of order 2 or higher. It follows that the residue of (2.35) at infinity vanishes, and hence

$$\frac{1}{2\pi i}\int_L \frac{1}{\zeta - z}\sum_{k=0}^{n} G_k(\zeta)\,d\zeta = 0,$$

which implies

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_L \frac{1}{\zeta - z}\Big[f(\zeta) - \sum_{k=0}^{n} G_k(\zeta)\Big]\,d\zeta. \tag{2.36}$$

But the function

$$f(\zeta) - \sum_{k=0}^{n} G_k(\zeta) \tag{2.37}$$

is analytic on $\overline{I(L)}$, since "all the poles of $f(z)$ have been subtracted out." Therefore we can apply Cauchy's integral formula (Theorem I.14.1) to (2.37), obtaining

$$\frac{1}{2\pi i}\int_L \frac{1}{\zeta - z}\Big[f(\zeta) - \sum_{k=0}^{n} G_k(\zeta)\Big]\,d\zeta = f(z) - \sum_{k=0}^{n} G_k(z). \tag{2.38}$$

Comparing (2.36) and (2.38), we again obtain formula (2.31).

THEOREM 2.7 (*Cauchy's theorem on partial fraction expansions*). *Let $f(z)$ be an analytic function whose only singular points in the finite plane are poles, and let $\{L_n\}$ be a sequence of closed rectifiable Jordan curves with the following properties*:

1. *The origin lies inside each curve L_n $(n = 1, 2, \ldots)$;*
2. *None of the curves passes through poles of $f(z)$;*
3. *Each curve in the sequence is contained in the next, i.e., $L_n \subset I(L_{n+1})$;*
4. *If r_n is the distance from the origin to L_n, then*

$$\lim_{n\to\infty} r_n = \infty. \tag{2.39}$$

Let the poles of $f(z)$ be ordered in such a way that L_n contains the first $nt_n + 1$ poles $b_0 = 0, b_1, \ldots, b_{m_n}$, with principal parts $G_0(z), G_1(z), \ldots, G_{m_n}(z)$, so that $m_n \leqslant m_{n+1}$ $(n = 1, 2, \ldots)$.[7] *Moreover, suppose that*

$$\overline{\lim_{n\to\infty}} \int_{L_n} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\,ds = M < \infty \tag{2.40}$$

[7] As in the lemma, $b_0 = 0$ may not be a pole of $f(z)$, in which case $G_0(z) \equiv 0$ (cf. footnote 6).

for some integer $p \geqslant -1$, *where* $ds = |d\zeta|$, *and let* z *be an arbitrary regular point of* $f(z)$. *Then, if* $p = -1$,

$$f(z) = \lim_{n \to \infty} \sum_{k=0}^{m_n} G_k(z), \tag{2.41}$$

while, if $p > -1$,

$$f(z) = \lim_{n \to \infty} \sum_{k=0}^{m_n} [G_k(z) + P_k(z)], \tag{2.42}$$

where the $P_k(z)$ *are polynomials of degree no higher than* p, *and the convergence in both* (2.41) *and* (2.42) *is uniform on every compact set containing no poles of* $f(z)$.[8]

Proof. First suppose that $p = -1$, so that the condition (2.40) becomes simply

$$\overline{\lim_{n \to \infty}} \int_{L_n} |f(\zeta)|\, ds = M < \infty.$$

Then, if $|z| \leqslant R < r_n$, where r_n is the distance from the origin to L_n, we have

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{L_n} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right| &\leqslant \frac{1}{2\pi(r_n - R)} \int_{L_n} |f(\zeta)|\, ds \\ &\leqslant \frac{M}{2\pi(r_n - R)} \to 0 \qquad \text{as} \quad n \to \infty \end{aligned} \tag{2.43}$$

[cf. (2.39)]. Therefore, replacing L by L_n and n by m_n in formula (2.31) [the lemma obviously applies here], and letting $n \to \infty$, we obtain the expansion (2.41). Moreover, according to (2.43), the remainder term approaches zero uniformly on the disk $|z| \leqslant R$, and hence the convergence in (2.41) is uniform on every compact set containing no poles of $f(z)$.

Next, assuming that $p > -1$, we replace $(\zeta - z)^{-1}$ in formula (2.31) by the expansion

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \frac{1}{1 - \dfrac{z}{\zeta}} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \cdots + \frac{z^p}{\zeta^{p+1}} + \frac{1}{\zeta - z} \frac{z^{p+1}}{\zeta^{p+1}}.$$

Then (2.31) becomes

$$f(z) = \sum_{k=0}^{n} G_k(z) + \sum_{j=0}^{p} \frac{z^j}{2\pi i} \int_L \frac{f(\zeta)}{\zeta^{j+1}}\, d\zeta + \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\zeta - z} \frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta. \tag{2.44}$$

[8] We wish to exclude the possibility of any of the terms in (2.41) or (2.42) becoming infinite.

Observing that the poles of the function $f(\zeta)/\zeta^{j+1}$ inside L are the points $b_0, b_1, \ldots, b_n$, we write

$$\operatorname*{Res}_{\zeta = b_k} \frac{f(\zeta)}{\zeta^{j+1}} = A_k^{(j)},$$

where $j = 0, 1, \ldots, p$ and $k = 0, 1, \ldots, n$. It follows that

$$\sum_{j=0}^{p} \frac{z^j}{2\pi i} \int_L \frac{f(\zeta)}{\zeta^{j+1}}\, d\zeta = \sum_{j=0}^{p} (A_0^{(j)} + A_1^{(j)} + \cdots + A_n^{(j)}) z^j = \sum_{k=0}^{n} P_k(z),$$

where the polynomials

$$P_k(z) = A_k^{(0)} + A_k^{(1)} z + \cdots + A_k^{(p)} z^p \tag{2.45}$$

have degree no higher than p. Therefore we can write (2.44) in the form

$$f(z) = \sum_{k=0}^{n} [G_k(z) + P_k(z)] + \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\zeta - z} \frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta. \tag{2.46}$$

We now replace L by L_n and n by m_n in (2.46), and examine the remainder term

$$\frac{1}{2\pi i} \int_{L_n} \frac{f(\zeta)}{\zeta - z} \frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta.$$

If $|z| \leqslant R < r_n$, the analogue of the estimate (2.43) is

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{L_n} \frac{f(\zeta)}{\zeta - z} \frac{z^{p+1}}{\zeta^{p+1}}\, d\zeta \right| &\leqslant \frac{R^{p+1}}{2\pi(r_n - R)} \int_{L_n} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\, ds \\ &\leqslant \frac{MR^{p+1}}{2\pi(r_n - R)} \to 0 \qquad \text{as} \quad n \to \infty. \end{aligned} \tag{2.47}$$

Therefore, taking the limit of (2.46) as $n \to \infty$ (with L_n and m_n instead of L and n), we obtain the expansion (2.42). Moreover, according to (2.47), the remainder term approaches zero uniformly on the disk $|z| \leqslant R$, and hence the convergence in (2.42) is uniform on every compact set containing no poles of $f(z)$.

Remark 1. The expansions (2.41) and (2.42) are called *partial fraction expansions* of $f(z)$, by analogy with the familiar case where $f(z)$ is a rational function.[9] We can also write (2.42) in the form

$$f(z) = G_0(z) + P_0(z) + \sum_{n=0}^{m_n} \{[G_{m_n+1}(z) + P_{m_n+1}(z)] + \cdots + [G_{m_{n+1}}(z) + P_{m_{n+1}}(z)]\},$$

where $m_0 = 0$.

[9] See e.g., G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, revised edition, The Macmillan Co., New York (1959), p. 84.

Remark 2. Suppose that besides the properties of the sequence $\{L_n\}$ listed in the statement of Theorem 2.7, there exists a constant $\lambda > 0$ such that

$$\frac{l_n}{r_n} \leqslant \lambda \qquad (n = 1, 2, \ldots),$$

where l_n is the length of L_n.[10] Then, since

$$\int_{L_n} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\, ds \leqslant \max_{\zeta \in L_n} |f(\zeta)| \frac{l_n}{r_n^{p+1}} \leqslant \lambda \frac{\max\limits_{\zeta \in L_n} |f(\zeta)|}{r_n^p},$$

(2.40) is implied by the simpler condition

$$\overline{\lim_{n \to \infty}} \frac{\max\limits_{\zeta \in L_n} |f(\zeta)|}{r_n^p} < \infty, \tag{2.48}$$

which permits

$$\max_{\zeta \in L_n} |f(\zeta)|$$

to approach infinity as $n \to \infty$, but no faster than some integral power of r_n.

10. Examples of Partial Fraction Expansions

We now give some examples illustrating the use of Cauchy's theorem on partial fraction expansions, proved in the preceding section.

Example 1. First we consider the function $\sec z$. For the sequence of contours $\{L_n\}$ figuring in Theorem 2.7, we choose squares with centers at the origin and sides of length $2n\pi$ parallel to the coordinate axes. On the sides parallel to the imaginary axis,

$$z = \pm n\pi + iy,$$

and hence

$$|\sec z| = \frac{1}{|\cos(\pm n\pi + iy)|} = \frac{1}{|\cos iy|} = \frac{1}{\cosh y},$$

while on the sides parallel to the real axis,

$$z = x \pm in\pi,$$

and hence

$$|\sec z| = \frac{1}{|\cos(x \pm in\pi)|} \leqslant \frac{1}{\sinh n\pi}$$

[cf. (I.9.48)]. It follows that

$$\int_{L_n} |\sec \zeta|\, ds \leqslant 2 \int_{-n\pi}^{n\pi} \frac{dy}{\cosh y} + 4n\pi \frac{1}{\sinh n\pi}.$$

[10] For example, this is the case if the L_n are homothetic figures with ray center $z = 0$ (cf. Sec. I.33).

Since the integral

$$\int_{-\infty}^{\infty} \frac{dy}{\cosh y}$$

converges, and since

$$\frac{4n\pi}{\sinh n\pi} \to 0 \qquad \text{as} \quad n \to \infty,$$

the condition (2.40) is satisfied for $p = -1$, and we can use the expansion (2.41).

Inside L_n, the function

$$\sec z = \frac{1}{\cos z}$$

has poles at the points

$$z = (k - \tfrac{1}{2})\pi \qquad (-n + 1 \leqslant k \leqslant n),$$

where the poles are all simple, since the zeros of $\cos z$ are simple. Obviously,

$$\operatorname*{Res}_{z=(k-\frac{1}{2})\pi} \sec z = -\frac{1}{\sin (k - \frac{1}{2})\pi} = (-1)^k,$$

and therefore the principal part of $\sec z$ at the pole $z = (k - \frac{1}{2})\pi$ is

$$G_k(z) = \frac{(-1)^k}{z - (k - \frac{1}{2})\pi}.$$

Moreover, we have $G_0(z) \equiv 0$, since $z = 0$ is not a pole of $\sec z$. Therefore formula (2.41) gives

$$\begin{aligned} \sec z &= \lim_{n\to\infty} \sum_{k=-n+1}^{n} \frac{(-1)^k}{z - (k - \frac{1}{2})\pi} \\ &= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{(-1)^k}{z - (k - \frac{1}{2})\pi} + \sum_{k=-n+1}^{0} \frac{(-1)^k}{z - (k - \frac{1}{2})\pi} \right]. \end{aligned} \tag{2.49}$$

Replacing k by $1 - j$ in the second sum on the right, we find that

$$\sum_{k=-n+1}^{0} \frac{(-1)^k}{z - (k - \frac{1}{2})\pi} = \sum_{j=1}^{n} \frac{(-1)^j}{z + (j - \frac{1}{2})\pi},$$

which implies

$$\begin{aligned} \sec z &= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{(-1)^k}{z - (k - \frac{1}{2})\pi} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{z + (k - \frac{1}{2})\pi} \right] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^k \frac{(2k - 1)\pi}{z^2 - (k - \frac{1}{2})^2\pi^2}, \end{aligned}$$

after changing from j to k. Thus, finally, we have the expansion

$$\sec z = \sum_{k=1}^{\infty} (-1)^k \frac{(2k - 1)\pi}{z^2 - (k - \frac{1}{2})^2\pi^2}, \tag{2.50}$$

which converges uniformly on every compact set containing none of the points

$$z = (k - \tfrac{1}{2})\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Example 2. Next we consider the function $\cot z$, choosing the contours L_n to be squares with centers at the origin and sides of length $(2n + 1)\pi$ parallel to the coordinate axes. On the sides parallel to the imaginary axis,

$$z = \pm(n + \tfrac{1}{2})\pi + iy,$$

and hence

$$|\cot z| = \frac{\cos[\pm(n + \frac{1}{2})\pi + iy]}{\sin[\pm(n + \frac{1}{2})\pi + iy]} = \frac{|\sin(iy)|}{|\cos(iy)|} = \left|\frac{e^y - e^{-y}}{e^y + e^{-y}}\right| < 1,$$

while on the sides parallel to the real axis,

$$z = x \pm i(n + \tfrac{1}{2})\pi,$$

and hence

$$|\cot z| = \frac{\cos[x \pm i(n + \frac{1}{2})\pi]}{\sin[x \pm i(n + \frac{1}{2})\pi]} \leqslant \frac{\cosh(n + \frac{1}{2})\pi}{\sinh(n + \frac{1}{2})\pi}$$
$$= \frac{1 + e^{-(2n+1)\pi}}{1 - e^{-(2n+1)\pi}} \leqslant \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

[cf. (I.9.48)]. Therefore

$$|\cot z| \leqslant \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

on every square L_n. This means that the condition (2.48), and consequently the condition (2.40), is satisfied for $p = 0$, which allows us to use the expansion (2.42).

Inside L_n, the function

$$\cot z = \frac{\cos z}{\sin z}$$

has poles at the points

$$z = k\pi \qquad (-n \leqslant k \leqslant n).$$

These poles are all simple, since the zeros of $\sin z$ are simple. Clearly

$$\operatorname*{Res}_{z=k\pi} \cot z = \frac{\cos k\pi}{\cos k\pi} = 1,$$

and therefore the principal part of $\cot z$ at the pole $z = k\pi$ is

$$G_k(z) = \frac{1}{z - k\pi}.$$

In the present case, the polynomials $P_k(z)$ defined by (2.45) reduce to constants:

$$P_k(z) = A_k^{(0)} = \operatorname*{Res}_{\zeta=k\pi} \frac{\cot \zeta}{\zeta},$$

The function $\cot \zeta/\zeta$ is obviously even, and hence its Laurent expansion at the origin contains only even powers of ζ, which implies

$$\operatorname*{Res}_{\zeta=0} \frac{\cot \zeta}{\zeta} = 0.$$

Moreover, the points $\zeta = k\pi$ $(k \neq 0)$ are simple poles of $\cot \zeta/\zeta$, and hence

$$\operatorname*{Res}_{\zeta=k\pi} \frac{\cot \zeta}{\zeta} = \frac{\cos k\pi}{k\pi \cos k\pi} = \frac{1}{k\pi} \qquad (k \neq 0).$$

Thus

$$P_0(z) = 0, \qquad P_k(z) = \frac{1}{k\pi} \qquad (k \neq 0),$$

and formula (2.42) gives

$$\begin{aligned} \cot z &= \lim_{n\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{n} \left(\frac{1}{z - k\pi} + \frac{1}{k\pi}\right) + \sum_{k=1}^{n} \left(\frac{1}{z + k\pi} - \frac{1}{k\pi}\right)\right] \\ &= \lim_{n\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{n} \left(\frac{1}{z - k\pi} + \frac{1}{z + k\pi}\right)\right] \qquad (2.51) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z - k\pi} + \frac{1}{z + k\pi}\right) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2\pi^2}, \end{aligned}$$

where the convergence is uniform on every compact set containing none of the points

$$z = k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Integrating (2.51) term by term along an arbitrary rectifiable curve γ which starts at the origin and passes through none of the points $z = k\pi$ $(k \neq 0)$, we obtain

$$\int_0^z \left(\cot \zeta - \frac{1}{\zeta}\right) d\zeta = \sum_{k=1}^{\infty} \operatorname{Ln} \left(\frac{k\pi - z}{k\pi} \frac{k\pi + z}{k\pi}\right) = \sum_{k=1}^{\infty} \operatorname{Ln} \left(1 - \frac{z^2}{k^2\pi^2}\right), \qquad (2.52)$$

where the logarithms on the right have perfectly definite values, i.e., those corresponding to the values of the integral

$$\int_\gamma \left(\frac{1}{z - k\pi} + \frac{1}{z + k\pi}\right) dz$$

(cf. Sec. I.69). The integral on the left in (2.52) equals one of the values of the function

$$\operatorname{Ln} \frac{\sin z}{z}.$$

Therefore

$$\operatorname{Ln} \frac{\sin z}{z} = \sum_{k=1}^{\infty} \operatorname{Ln} \left(1 - \frac{z^2}{k^2\pi^2}\right),$$

which implies

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right),$$

or

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right)$$

(concerning infinite products, see Sec. I.77).

Example 3. The expansions of $\csc z$ and $\tan z$ can be derived directly from the expansions of $\sec z$ and $\cot z$ just found. First we consider $\csc z$. Since

$$\csc z = \sec\left(\tfrac{1}{2}\pi - z\right),$$

it follows from formula (2.49) that

$$\begin{aligned}\csc z &= \lim_{n\to\infty} \sum_{k=-n+1}^{n} \frac{(-1)^k}{\frac{1}{2}\pi - z - (k - \frac{1}{2})\pi} \\ &= \lim_{n\to\infty} \sum_{k=-n+1}^{n} \frac{(-1)^k}{-(k-1)\pi - z} = \lim_{n\to\infty} \sum_{k=-n+1}^{n} \frac{(-1)^{k-1}}{(k-1)\pi + z}.\end{aligned}$$

Changing $k - 1$ to k', and then dropping the prime, we have

$$\begin{aligned}\csc z &= \lim_{n\to\infty} \sum_{k=-n}^{n-1} \frac{(-1)^k}{z + k\pi} \\ &= \lim_{n\to\infty} \left[\frac{1}{z} - \left(\frac{1}{z-\pi} + \frac{1}{z+\pi}\right) + \cdots \right. \\ &\qquad \left. + (-1)^{n-1}\left(\frac{1}{z-(n-1)\pi} + \frac{1}{z+(n-1)\pi}\right) + (-1)^n \frac{1}{z - n\pi}\right].\end{aligned}$$

Adding the term

$$(-1)^n \frac{1}{z + n\pi} \tag{2.53}$$

to the expression in brackets, where (2.53) approaches zero uniformly on every compact set as $n \to \infty$, we obtain the desired partial fraction expansion

$$\begin{aligned}\csc z &= \lim_{n\to\infty} \left[\frac{1}{z} - \frac{2z}{z^2 - \pi^2} + \frac{2z}{z^2 - 4\pi^2} - \cdots + (-1)^n \frac{2z}{z^2 - n^2\pi^2}\right] \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^k \frac{2z}{z^2 - k^2\pi^2}.\end{aligned}$$

Similarly, in the case of $\tan z$, we start from the formula

$$\tan z = \cot\left(\tfrac{1}{2}\pi - z\right),$$

which together with (2.51) implies

$$\tan z = \lim_{n\to\infty}\left[\frac{1}{\frac{1}{2}\pi - z} + \sum_{k=1}^{n}\left(\frac{1}{\frac{1}{2}\pi - z - k\pi} + \frac{1}{\frac{1}{2}\pi - z + k\pi}\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{\frac{1}{2}\pi - z} + \frac{1}{-\frac{1}{2}\pi - z}\right) + \left(\frac{1}{\frac{3}{2}\pi - z} + \frac{1}{-\frac{3}{2}\pi - z}\right)\right.$$

$$\left. + \cdots + \left(\frac{1}{(n-\frac{1}{2})\pi - z} + \frac{1}{-(n-\frac{1}{2})\pi - z}\right) + \frac{1}{(n+\frac{1}{2})\pi - z}\right].$$

Subtracting the term

$$\frac{1}{(n+\frac{1}{2})\pi - z} \tag{2.54}$$

from the expression in brackets, where (2.54) approaches zero uniformly on every compact set as $n \to \infty$, we obtain the desired partial fraction expansion

$$\tan z = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{-2z}{z^2 - (k-\frac{1}{2})^2\pi^2} = \sum_{k=1}^{\infty}\frac{2z}{(k-\frac{1}{2})^2\pi^2 - z^2}.$$

Example 4. Finally we use the method of partial fraction expansions to derive the Taylor series expansions of the functions

$$\sec z \quad \text{and} \quad \cot z - \frac{1}{z}$$

at the origin. These Taylor series were already found in Sec. I.90 by an entirely different method, i.e., by division of power series. As we shall see, a comparison of the results of the two methods allows us to evaluate some interesting numerical series.

First we consider the function $\sec z$, which is analytic on the disk $K: |z| < \pi/2$. As shown in Example 1, $\sec z$ has the partial fraction expansion

$$\sec z = \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)\pi}{z^2 - (k-\frac{1}{2})^2\pi^2} = \sum_{k=1}^{\infty} f_k(z),$$

where

$$f_k(z) = (-1)^k\left[\frac{1}{z - (k-\frac{1}{2})\pi} - \frac{1}{z + (k-\frac{1}{2})\pi}\right]$$

[cf. (2.50)]. This series is uniformly convergent on every compact set containing no poles of $\sec z$, in particular, on every compact subset of the disk K, and each of its terms is obviously analytic on K. Therefore, according to Theorem I.18.1 and the subsequent remark, the coefficient of a given power of z in the Taylor series of $\sec z$ on K is obtained by adding all the coefficients of the same power of z in the separate Taylor series of the functions $f_k(z)$ on K.

It is easy to see that

$$\begin{aligned} f_k(z) &= -(-1)^k\left[\sum_{n=0}^{\infty}\frac{z^n}{[(k-\frac{1}{2})\pi]^{n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{[(k-\frac{1}{2})\pi]^{n+1}}\right] \\ &= 2(-1)^{k-1}\sum_{m=0}^{\infty}\frac{z^{2m}}{[(k-\frac{1}{2})\pi]^{2m+1}} \end{aligned} \tag{2.55}$$

if $|z| < (k-\frac{1}{2})\pi$. Thus, on the disk $K: |z| < \pi/2$, the expansions (2.55) are valid for all the $f_k(z)$, $k = 1, 2, \ldots$. It is apparent from (2.55) that the coefficients of odd powers of z in the Taylor series of $f_k(z)$ vanish, while the coefficients of z^{2m} equal

$$\frac{2(-1)^{k-1}}{[(k-\frac{1}{2})\pi]^{2m+1}} = 2\left(\frac{2}{\pi}\right)^{2m+1}\frac{(-1)^{k-1}}{(2k-1)^{2m+1}} \qquad (m = 0, 1, 2, \ldots).$$

Therefore the coefficients of odd powers of z in the Taylor series of $\sec z$ on K vanish (which could have been predicted from the fact that $\sec z$ is an even function), while the coefficients of z^{2m} can be written as series

$$2\left(\frac{2}{\pi}\right)^{2m+1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{2m+1}} \qquad (m = 0, 1, 2, \ldots).$$

It follows that

$$\sec z = \sum_{m=0}^{\infty}\left[2\left(\frac{2}{\pi}\right)^{2m+1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{2m+1}}\right]z^{2m} \qquad \left(|z| < \frac{\pi}{2}\right). \tag{2.56}$$

Recalling the somewhat different appearing expansion

$$\sec z = \sum_{m=0}^{\infty}(-1)^m\frac{E_{2m}}{(2m)!}z^{2m} \qquad \left(|z| < \frac{\pi}{2}\right) \tag{2.57}$$

derived previously [see (I.18.37)], which involves the *Euler numbers* E_{2m} ($E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61, \ldots$), and comparing (2.56) and (2.57), we find that

$$2\left(\frac{2}{\pi}\right)^{2m+1}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{2m+1}} = (-1)^m\frac{E_{2m}}{(2m)!} \qquad (m = 0, 1, 2, \ldots).$$

In particular, we have the interesting formulas

$$\begin{aligned} 1+\frac{1}{3}+\frac{1}{5}+\cdots &= \frac{E_0}{2}\frac{\pi}{2} = \frac{\pi}{4}, \\ 1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots &= -\frac{E_2}{2\cdot 2!}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{32}, \\ 1+\frac{1}{3^5}+\frac{1}{5^5}+\cdots &= \frac{E_4}{2\cdot 4!}\left(\frac{\pi}{2}\right)^5 = \frac{5\pi^5}{1536}, \end{aligned}$$

and so on.

Next we turn to the function $\cot z - z^{-1}$, which is analytic on the disk $|z| < \pi$. In this case, we start from the partial fraction expansion (2.51), which gives

$$\cot z - \frac{1}{z} = \sum_{k=1}^{\infty} f_k(z),$$

where now

$$f_k(z) = \frac{1}{z - k\pi} + \frac{1}{z + k\pi} = -\sum_{n=0}^{\infty} \frac{z^k}{(k\pi)^{n+1}} + \sum_{n=0}^{\infty} (-1)^k \frac{z^k}{(k\pi)^{n+1}}$$
$$= -2 \sum_{m=1}^{\infty} \frac{z^{2m-1}}{(k\pi)^{2m}} \qquad (|z| < k\pi).$$

Therefore the coefficients of even powers of z in the Taylor series of $\cot z - z^{-1}$ vanish (which could have been predicted from the fact that $\cot z - z^{-1}$ is an odd function), while the coefficients of z^{2m-1} can be written as series

$$-2 \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{2m}} = -\frac{2}{\pi^{2m}} \sum_{k=1}^{\infty} \frac{1}{k^{2m}} \qquad (m = 1, 2, \ldots).$$

It follows that

$$\cot z - \frac{1}{z} = \sum_{m=1}^{\infty} \left[-\frac{2}{\pi^{2m}} \sum_{n=1}^{\infty} \frac{1}{k^{2m}} \right] z^{2m-1}. \tag{2.58}$$

Recalling the somewhat different appearing expansion

$$\cot z - \frac{1}{z} = \sum_{m=1}^{\infty} (-1)^m \frac{2^{2m} B_{2m}}{(2m)!} z^{2m-1} \tag{2.59}$$

derived previously [see (I.18.32)], which involves the *Bernoulli numbers* B_{2n}, and comparing (2.58) and (2.59), we find that

$$\frac{2}{\pi^{2m}} \sum_{k=1}^{\infty} \frac{1}{k^{2m}} = (-1)^{m-1} \frac{2^{2m} B_{2m}}{(2m)!}. \tag{2.60}$$

Since the left-hand side of (2.60) is positive, the same must be true of the right-hand side, i.e., $(-1)^{m-1} B_{2m} > 0$. This means that the Bernoulli numbers B_{2m} $(m = 1, 2, \ldots)$ must have alternating signs (it will be recalled from Sec. I.90 that $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}, \ldots$). In particular, (2.60) implies the interesting formulas

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{B_2}{2 \cdot 2!} (2\pi)^2 = \frac{\pi^2}{6},$$
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = -\frac{B_4}{2 \cdot 4!} (2\pi)^4 = \frac{\pi^4}{90},$$
$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{B_6}{2 \cdot 6!} (2\pi)^6 = \frac{\pi^6}{945},$$

and so on.

*11. Interpolation Theory

Given a set of m points

$$z_1, z_2, \ldots, z_m \tag{2.61}$$

belonging to a domain G, and a corresponding set of m positive integers

$$\alpha_1, \alpha_2, \ldots, \alpha_m \tag{2.62}$$

such that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_m = n, \tag{2.63}$$

we look for a polynomial $P(z)$ satisfying the conditions

$$P(z_k) = f(z_k), \ldots, P^{(\alpha_k - 1)}(z_k) = f^{(\alpha_k - 1)}(z_k) \qquad (k = 1, 2, \ldots, m), \tag{2.64}$$

where $f(z)$ is analytic on G. Any polynomial satisfying these conditions will be called an *interpolation polynomial* for $f(z)$, corresponding to the *interpolation points* (2.61), with *orders* (2.62). If $P(z)$ is such a polynomial, then the difference

$$R(z) = f(z) - P(z)$$

is obviously analytic on G, and satisfies the conditions

$$R(z_k) = \cdots = R^{(\alpha_k - 1)}(z_k) = 0 \qquad (k = 1, 2, \ldots, m).$$

Therefore, at every point z_k $(k = 1, 2, \ldots, m)$, $R(z)$ has a zero of order no less than α_k. Let $P_1(z)$ be another polynomial satisfying the conditions (2.64). Then the corresponding function

$$R_1(z) = f(z) - P_1(z)$$

also has a zero of order α_k at every point z_k, and the same is true of the function

$$R(z) - R_1(z) = P(z) - P_1(z).$$

In other words, $P(z) - P_1(z)$ is a polynomial with at least n zeros $z_1, z_2, \ldots, z_m$,[11] of orders no less than $\alpha_1, \alpha_2, \ldots, \alpha_m$, which means that $P(z) - P_1(z)$ is divisible by the polynomial

$$\omega(z) = (z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2} \cdots (z - z_m)^{\alpha_m} \tag{2.65}$$

of degree n. Therefore, if the degree of each of the polynomials $P(z)$ and $P_1(z)$ is less than n, their difference $P(z) - P_1(z)$ must vanish identically, i.e., *a polynomial of degree less than n satisfying the conditions* (2.64) *must be unique.*

---

[11] Recall (2.63) and the fact that a zero of order α_k is counted as α_k zeros.

We now show that such an interpolation polynomial always exists:

THEOREM 2.8. *Let L be a closed rectifiable Jordan curve contained in G, such that all the interpolation points $z_1, z_2, \ldots, z_m$ belong to $I(L)$, and let $\omega(z)$ be the polynomial* (2.65). *Then the integral*

$$P(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega(\zeta)}\,\frac{\omega(\zeta) - \omega(z)}{\zeta - z}\,d\zeta \tag{2.66}$$

is a polynomial of degree less than n satisfying the conditions (2.64).

Proof. Writing

$$\omega(z) = \sum_{j=0}^{n} A_j z^j,$$

we see that

$$\begin{aligned}\frac{\omega(\zeta) - \omega(z)}{\zeta - z} &= \frac{\sum_{k=0}^{n} A_j(\zeta^j - z^j)}{\zeta - z}\\ &= A_1 + A_2(\zeta + z) + \cdots + A_n(\zeta^{n-1} + \cdots + z^{n-1})\\ &= A_1 + A_2\zeta + \cdots + A_n\zeta^{n-1}\\ &\quad + (A_2 + A_3\zeta + \cdots + A_n\zeta^{n-2})z + \cdots + A_n z^{n-1}\\ &= B_{n-1}(\zeta) + B_{n-2}(\zeta)\,z + \cdots + B_0(\zeta)\,z^{n-1},\end{aligned}$$

where $B_{n-1}(\zeta), B_{n-2}(\zeta), \ldots, B_0(\zeta)$ are polynomials in ζ, with degrees the same as their subscripts. Therefore $P(z)$ can be represented in the form

$$\begin{aligned}P(z) &= \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega(\zeta)}\sum_{j=0}^{n-1} B_{n-1-j}(\zeta)\,z^j\,d\zeta\\ &= \sum_{j=0}^{n-1}\left[\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega(\zeta)}\,B_{n-1-j}(\zeta)\,d\zeta\right]z^j = \sum_{j=0}^{n-1} C_j z^j,\end{aligned}$$

where

$$C_j = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega(\zeta)}\,B_{n-1-j}(\zeta)\,d\zeta \qquad (j = 0, 1, \ldots, n-1).$$

Thus (2.66) represents a polynomial of degree less than n.

We now form the difference

$$R(z) = f(z) - P(z).$$

If $z \in I(L)$, then, according to Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

and hence

$$R(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega(\zeta)}\,\frac{\omega(\zeta) - \omega(z)}{\zeta - z}\,d\zeta,$$

so that

$$R(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\frac{\omega(z)}{\omega(\zeta)}\,d\zeta = \frac{\omega(z)}{2\pi i}\int_L \frac{f(\zeta)}{\omega(\zeta)(\zeta - z)}\,d\zeta. \qquad (2.67)$$

The integral

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega(\zeta)(\zeta - z)}\,d\zeta = \frac{1}{2\pi i}\int_L \frac{f(\zeta)/\omega(\zeta)}{\zeta - z}\,d\zeta$$

is obviously an integral of the Cauchy type, and hence represents a function analytic on $I(L)$. Therefore

$$\frac{R(z)}{(z - z_k)^{\alpha_k}} = (z - z_1)^{\alpha_1}\cdots(z - z_{k-1})^{\alpha_{k-1}} \times (z - z_{k+1})^{\alpha_{k+1}}\cdots(z - z_m)^{\alpha_m}\frac{1}{2\pi i}\int_L \frac{f(\zeta)/\omega(\zeta)}{\zeta - z}\,d\zeta$$

is analytic at z_k [in fact, on all of $I(L)$], so that $R(z)$ has a zero at z_k of order α_k or greater. In other words,

$$R(z_k) = \cdots = R^{(\alpha_k - 1)}(z_k) = 0 \qquad (k = 1, 2, \ldots, m),$$

which immediately implies that (2.66) satisfies the conditions (2.64), as asserted.

Remark. Clearly, (2.66) must be the *unique* polynomial of degree less than n satisfying (2.64). Formula (2.66) for the interpolation polynomial, and formula (2.67) for the *remainder term* in the *interpolation formula*

$$f(z) = P(z) + R(z),$$

are known as *Hermite's formulas.*

We now give some examples illustrating Theorem 2.8.

Example 1. First let $m = 1$, so that there is only one interpolation point z_1, of order $\alpha_1 = n$. Then the problem reduces to finding a polynomial $P(z) = \mathscr{T}_{n-1}(z)$ of degree no greater than $n - 1$ which satisfies the conditions

$$\mathscr{T}_{n-1}(z_1) = f(z_1), \mathscr{T}'_{n-1}(z_1) = f'(z_1), \ldots, \mathscr{T}^{(n-1)}_{n-1}(z_1) = f^{(n-1)}(z_1).$$

In the present case,

$$\omega(z) = \omega_n(z) = (z - z_1)^n,$$

and formula (2.66) gives

$$\mathscr{T}_{n-1}(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{(\zeta - z_1)^n}\frac{(\zeta - z_1)^n - (z - z_1)^n}{\zeta - z}\,d\zeta. \qquad (2.68)$$

Since

$$\frac{1}{(\zeta - z_1)^n}\frac{(\zeta - z_1)^n - (z - z_1)^n}{\zeta - z}$$
$$= \frac{1}{(\zeta - z_1)^n}[(\zeta - z_1)^{n-1} + (\zeta - z_1)^{n-2}(z - z_1) + \cdots + (z - z_1)^{n-1}]$$
$$= \frac{1}{\zeta - z_1} + \frac{1}{(\zeta - z_1)^2}(z - z_1) + \cdots + \frac{1}{(\zeta - z_1)^n}(z - z_1)^{n-1}, \tag{2.69}$$

we can write (2.68) in the form

$$\mathscr{T}_{n-1}(z) = \sum_{k=0}^{n-1}\left[\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{(\zeta - z_1)^k}\,d\zeta\right](z - z_0)^k = \sum_{k=0}^{n-1}\frac{f^{(k)}(z_1)}{k!}(z - z_1)^k,$$

which is just *Taylor's interpolation polynomial* of degree $n - 1$, i.e., the $(n - 1)$th partial sum of the Taylor series of $f(z)$ [of course, this is the anticipated result].

Correspondingly, formula (2.67) reduces to

$$R(z) = R_n(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\frac{(z - z_1)^n}{(\zeta - z_1)^n}\,d\zeta, \tag{2.70}$$

which is the *remainder term in Taylor's formula.* Because of Cauchy's integral theorem for a system of contours (Theorem I.13.4), the value of (2.70) does not change if we replace L by any circle $\gamma_\rho: |z - z_1| = \rho$, where ρ is less than the distance Δ from the point z_1 to the boundary of the domain G on which $f(z)$ is analytic. Thus, for points z inside γ_ρ, we have

$$R_n(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}\frac{(z - z_1)^n}{(\zeta - z_1)^n}\,d\zeta,$$

and hence

$$|R_n(z)| \leqslant \frac{1}{2\pi}\frac{M(\rho)}{\rho - |z - z_1|}\left(\frac{|z - z_1|}{\rho}\right)^n,$$

where

$$M(\rho) = \max_{\zeta \in \gamma_\rho} |f(\zeta)|.$$

Since $|z - z_1| < \rho$, we see that $|R_n(z)| \to 0$ as $n \to \infty$, which gives another proof of the fact that the sequence of Taylor polynomials $\mathscr{T}_{n-1}(z)$ converges to $f(z)$ on the largest disk $|z - z_1| < \Delta$ contained in the domain of analyticity of $f(z)$ [cf. Theorem I.16.7].

Example 2. Next let $m = n$, so that there are n interpolation points $z_1, \ldots, z_n$, each of order 1 ($\alpha_1 = \cdots = \alpha_n = 1$). The problem now reduces to finding a polynomial $P(z) = \mathscr{L}_{n-1}(z)$ of degree no greater than $n - 1$ which satisfies the conditions

$$\mathscr{L}_{n-1}(z_1) = f(z_1), \ldots, \mathscr{L}_{n-1}(z_n) = f(z_n).$$

In this case

$$\omega(z) = \omega_n(z) = (z - z_1)\cdots(z - z_n),$$

and formula (2.66) gives

$$\mathscr{L}_{n-1}(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_n(\zeta)}\,\frac{\omega_n(\zeta) - \omega_n(z)}{\zeta - z}\,d\zeta. \tag{2.71}$$

The simplest way to evaluate this integral is to use the residue theorem. The integrand $\varphi(\zeta)$, regarded as a function of ζ, has simple poles at the points $\zeta = z_k$ $(k = 1, \ldots, n)$, but no singular point at $\zeta = z$, since the polynomial $\omega_n(\zeta) - \omega_n(z)$ is divisible by $\zeta - z$. For the residues of $\varphi(\zeta)$ at the poles z_k we have the formula

$$\operatorname*{Res}_{\zeta = z_k} \varphi(\zeta) = \frac{f(z_k)}{\omega_n'(z_k)}\,\frac{\omega_n(z)}{z - z_k},$$

where

$$\frac{\omega_n(z)}{z - z_k} = (z - z_1)\cdots(z - z_{k-1})(z - z_{k+1})\cdots(z - z_n)$$

is a polynomial in z of degree $n - 1$. Therefore we obtain

$$\mathscr{L}_{n-1}(z) = \sum_{k=1}^{n} \operatorname*{Res}_{\zeta = z_k} \varphi(\zeta) = \sum_{k=1}^{n} \frac{f(z_k)}{\omega_n'(z_k)}\,\frac{\omega_n(z)}{z - z_k}, \tag{2.72}$$

an expression known as *Lagrange's interpolation polynomial* of degree $n - 1$. The corresponding remainder term is

$$R(z) = R_n(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\,\frac{\omega_n(z)}{\omega_n(\zeta)}, \tag{2.73}$$

whose behavior as $n \to \infty$ depends on the relation between the distribution of interpolation points $z_1, \ldots, z_n$ and the character of the function $f(z)$.

Example 3. To derive another representation of Lagrange's interpolation polynomial, we examine the expression

$$\sigma_n = \frac{1}{\omega_n(\zeta)}\,\frac{\omega_n(\zeta) - \omega_n(z)}{\zeta - z}$$

appearing in formula (2.71). Clearly, σ_n is a polynomial of degree $n - 1$ in z, which for $n = 1$ equals

$$\sigma_1 = \frac{1}{\zeta - z_1}\,\frac{(\zeta - z_1) - (z - z_1)}{\zeta - z} = \frac{1}{\zeta - z_1}.$$

Consider the difference

$$\delta_k = \sigma_{k+1} - \sigma_k = \frac{1}{\omega_{k+1}(\zeta)}\,\frac{\omega_{k+1}(\zeta) - \omega_{k+1}(z)}{\zeta - z} - \frac{1}{\omega_k(\zeta)}\,\frac{\omega_k(\zeta) - \omega_k(z)}{\zeta - z}.$$

Since

$$\omega_{k+1}(\zeta) = (\zeta - z_{k+1})\omega_k(\zeta), \quad \omega_{k+1}(z) = (z - z_{k+1})\omega_k(z),$$

we can represent δ_k in the form

$$\delta_k = \frac{(\zeta - z_{k+1})\omega_k(\zeta) - (z - z_{k+1})\omega_k(z) - (\zeta - z_{k+1})\omega_k(\zeta) + (\zeta - z_{k+1})\omega_k(z)}{(\zeta - z)\omega_{k+1}(\zeta)}$$

$$= \frac{(\zeta - z)\omega_k(z)}{(\zeta - z)\omega_{k+1}(\zeta)} = \frac{\omega_k(z)}{\omega_{k+1}(\zeta)}.$$

Setting

$$\delta_0 = \sigma_1 = \frac{1}{\zeta - z_1} = \frac{1}{\omega_1(\zeta)},$$

we calculate the sum

$$\sum_{k=0}^{n-1} \delta_k = \sigma_1 + \sum_{k=1}^{n-1} (\sigma_{k+1} - \sigma_k) = \sigma_n = \frac{1}{\omega_1(\zeta)} + \frac{\omega_1(z)}{\omega_2(\zeta)} + \cdots + \frac{\omega_{n-1}(z)}{\omega_n(\zeta)}.$$

Recalling the meaning of σ_n, we have

$$\frac{1}{\omega_n(\zeta)} \frac{\omega_n(\zeta) - \omega_n(z)}{\zeta - z} = \frac{1}{\omega_1(\zeta)} + \frac{\omega_1(z)}{\omega_2(\zeta)} + \cdots + \frac{\omega_{n-1}(z)}{\omega_n(\zeta)}, \tag{2.74}$$

which reduces to the identity (2.69) if we set $z_1 = \cdots = z_n$. Comparing (2.71) and (2.74), we find that

$$\mathscr{L}_{n-1}(z) = \sum_{k=0}^{n-1} \left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\omega_{k+1}(\zeta)} d\zeta\right] \omega_k(z). \tag{2.75}$$

When written in this form, $\mathscr{L}_{n-1}(z)$ is called *Newton's interpolation polynomial* of degree $n - 1$. The advantage of (2.75) over (2.72) is apparent when we go from $\mathscr{L}_{n-1}(z)$ to $\mathscr{L}_n(z)$: In (2.75) we need only add a new term

$$\left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\omega_{n+1}(\zeta)} d\zeta\right] \omega_n(z),$$

while in (2.72) we must not only add a new term but change all the old terms.

In the case where the sequence $\{\mathscr{L}_{n-1}(z)\}$ converges to $f(z)$, i.e.,

$$\lim_{n \to \infty} \mathscr{L}_{n-1}(z) = f(z),$$

formula (2.75) leads to the series

$$f(z) = \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\omega_{k+1}(\zeta)} d\zeta\right] \omega_k(z), \tag{2.76}$$

called *Newton's interpolation series*. If $z_1 = \cdots = z_n = \cdots$, then $\omega_{k+1}(\zeta)$ reduces to $(\zeta - z_1)^{k+1}$ and $\omega_k(z)$ to $(z - z_1)^k$, so that (2.76) contains the Taylor series

$$f(z) = \sum_{k=0}^{\infty} \left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^{k+1}} d\zeta\right] (z - z_1)^k \tag{2.77}$$

as a special case. However, it should be noted that whereas we have established conditions for convergence of (2.77), the problem of convergence of Newton's series, i.e., the investigation of conditions under which the remainder term in Newton's series [identical with the remainder term (2.73)] converges to zero as $n \to \infty$, requires a special investigation taking account of the distribution of the interpolation points and the behavior of the function $f(z)$.[12]

As just noted, when Newton's series reduces to Taylor's series, its coefficients

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_{k+1}(\zeta)}\,d\zeta \tag{2.78}$$

become simply

$$\frac{f^{(k)}(z_1)}{k!},$$

and can therefore be expressed in terms of derivatives of $f(z)$. In the general case, these coefficients can be expressed in terms of certain "finite-difference analogues" of derivatives, defined as follows:

DEFINITION. *By the first-order divided difference of $f(z)$ with respect to the two points z_1 and z_2, we mean the quantity*

$$\Delta^{(1)}[f(z); z_1, z_2] = \frac{f(z_2) - f(z_1)}{z_2 - z_1}. \tag{2.79}$$

Then the k'th-order divided difference of $f(z)$ with respect to the $k+1$ points $z_1, \ldots, z_{k+1}$ is defined inductively by the formula

$$\Delta^{(k)}[f(z); z_1, \ldots, z_{k+1}] = \frac{\Delta^{(k-1)}[f(z); z_2, \ldots, z_{k+1}] - \Delta^{(k-1)}[f(z); z_1, \ldots, z_k]}{z_{k+1} - z_1} \quad (k = 2, 3, \ldots). \tag{2.80}$$

It is now an easy matter to show that

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_{k+1}(\zeta)}\,d\zeta = \Delta^{(k)}[f(z); z_1, \ldots, z_{k+1}] \qquad (k = 1, 2, \ldots). \tag{2.81}$$

In fact, as might be expected, the proof is itself by induction. First we note that

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_2(\zeta)}\,d\zeta = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{(\zeta - z_1)(\zeta - z_2)}\,d\zeta = \frac{f(z_1)}{z_1 - z_2} + \frac{f(z_2)}{z_2 - z_1}$$
$$= \frac{f(z_2) - f(z_1)}{z_2 - z_1} = \Delta^{(1)}[f(z); z_1, z_2],$$

[12] Concerning this problem and interpolation theory in general, see the books by Gelfond, Nörlund and Whittaker, cited in the Bibliography on pp. 326–327. For an elementary result along these lines, see Problem 2.30.

where we have used the residue theorem[13] and the definition (2.79). This proves (2.81) for $k = 1$. Next suppose (2.81) has been proved for $k = m - 1$, so that

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_m(\zeta)}\,d\zeta = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{(\zeta - z_1)\cdots(\zeta - z_m)}\,d\zeta = \Delta^{(m-1)}[f(z); z_1, \ldots, z_m]. \tag{2.82}$$

Then

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{(\zeta - z_2)\cdots(\zeta - z_{m+1})}\,d\zeta = \Delta^{(m-1)}[f(z); z_2, \ldots, z_{m+1}], \tag{2.83}$$

where we have replaced the m points $z_1, \ldots, z_m$ by the m points $z_2, \ldots, z_{m+1}$. Forming the mth-order divided difference called for by the definition (2.80), and substituting from (2.82) and (2.83), we obtain

$$\begin{aligned}
&\Delta^{(m)}[f(z); z_1, \ldots, z_{m+1}] \\
&= \frac{\Delta^{(m-1)}[f(z); z_2, \ldots, z_{m+1}] - \Delta^{(m-1)}[f(z); z_1, \ldots, z_m]}{z_{m+1} - z_1} \\
&= \frac{1}{z_{m+1} - z_1}\frac{1}{2\pi i}\int_L f(\zeta)\left[\frac{1}{(\zeta - z_2)\cdots(\zeta - z_{m+1})} - \frac{1}{(\zeta - z_1)\cdots(\zeta - z_m)}\right]d\zeta \\
&= \frac{1}{z_{m+1} - z_1}\frac{1}{2\pi i}\int_L f(\zeta)\frac{(\zeta - z_1) - (\zeta - z_{m+1})}{(\zeta - z_1)\cdots(\zeta - z_{m+1})}\,d\zeta = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_{m+1}(\zeta)}\,d\zeta.
\end{aligned}$$

This is just formula (2.81) for $k = m$, and hence the inductive proof of (2.81) is complete. Finally, comparing (2.78) and (2.81), we find that

$$\mathscr{L}_{n-1}(z) = \sum_{k=0}^{n-1} \Delta^{(k)}[f(z); z_1, \ldots, z_{k+1}]\omega_k(z),$$

where we set

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_1(\zeta)}\,d\zeta = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z_1}\,d\zeta = f(z_1) = \Delta^{(0)}[f(z); z_1],$$

by definition.[14] In the same way, we see that Newton's interpolation series can be written in the form

$$\sum_{k=0}^{\infty} \Delta^{(k)}[f(z); z_1, \ldots, z_{k+1}]\omega_k(z).$$

---

[13] More generally,

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\omega_{k+1}(\zeta)}\,d\zeta = \sum_{j=1}^{k+1} \operatorname*{Res}_{\zeta = z_j} \frac{f(\zeta)}{\omega_{k+1}(\zeta)} = \sum_{j=1}^{k+1} \frac{f(z_j)}{\omega'_{k+1}(z_j)},$$

where

$$\omega'_{k+1}(z_j) = (z_j - z_1)\cdots(z_j - z_{j-1})(z_j - z_{j+1})\cdots(z_j - z_{k+1}).$$

[14] Note that in the definition of $\Delta^{(k)}[f(z); z_1, \ldots, z_{k+1}]$ and in the proof of (2.81) which were both by induction, we could have started (rather artificially) with the "zeroth-order divided difference" $\Delta^{(0)}[f(z); z_1] = f(z_1)$.

Example 4. Finally we consider the case of m interpolation points $z_1, \ldots, z_m$, all of the same order $\alpha_1 = \cdots = \alpha_m = p$, where later we will make p arbitrarily large, without changing the interpolation points themselves. The problem now reduces to finding a polynomial $\mathscr{J}_{mp-1}(z)$ of degree $mp - 1$ satisfying the conditions

$$\mathscr{J}_{mp-1}(z_k) = f(z_k),\ \mathscr{J}'_{mp-1}(z_k) = f'(z_k), \ldots, \mathscr{J}^{(p-1)}_{mp-1}(z_k) = f^{(p-1)}(z_k) \qquad (k = 1, 2, \ldots, m).$$

Writing

$$q(z) = (z - z_1)\cdots(z - z_m), \tag{2.84}$$

we have

$$\omega(z) = \omega_{mp}(z) = (z - z_1)^p \cdots (z - z_m)^p = [q(z)]^p,$$

and hence

$$\mathscr{J}_{mp-1}(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{[q(\zeta)]^p}\,\frac{[q(\zeta)]^p - [q(z)]^p}{\zeta - z}\,d\zeta. \tag{2.85}$$

Using the identity

$$\begin{aligned}\frac{1}{[q(\zeta)]^p}\,\frac{[q(\zeta)]^p - [q(z)]^p}{\zeta - z} &= \frac{1}{[q(\zeta)]^p}\,\frac{q(\zeta) - q(z)}{\zeta - z}\{[q(\zeta)]^{p-1} + [q(\zeta)]^{p-2}q(z) + \cdots + [q(z)]^{p-1}\} \\ &= \frac{q(\zeta) - q(z)}{\zeta - z}\left\{\frac{1}{q(\zeta)} + \frac{q(z)}{[q(\zeta)]^2} + \cdots + \frac{[q(z)]^{p-1}}{[q(\zeta)]^p}\right\},\end{aligned}$$

we transform (2.85) into

$$\mathscr{J}_{mp-1}(z) = \sum_{k=0}^{p-1}\left\{\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{[q(\zeta)]^{k+1}}\,\frac{q(\zeta) - q(z)}{\zeta - z}\,d\zeta\right\}[q(z)]^k. \tag{2.86}$$

Next we replace $\omega_n(z)$ by $q(z)$ in formula (2.74), obtaining

$$\begin{aligned}\frac{1}{[q(\zeta)]^{k+1}}\,\frac{q(\zeta) - q(z)}{\zeta - z} &= \frac{1}{[q(\zeta)]^k}\left[\frac{1}{q(\zeta)}\,\frac{q(\zeta) - q(z)}{\zeta - z}\right] \\ &= \frac{1}{[q(\zeta)]^k}\left[\frac{1}{\zeta - z_1} + \frac{z - z_1}{(\zeta - z_1)(\zeta - z_2)} + \cdots + \frac{(z - z_1)\cdots(z - z_{m-1})}{(\zeta - z_1)\cdots(\zeta - z_m)}\right].\end{aligned}$$

Therefore the coefficients in (2.86) become[15]

$$\begin{aligned}&\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{[q(\zeta)]^{k+1}}\,\frac{q(\zeta) - q(z)}{\zeta - z}\,d\zeta \\ &\quad = \sum_{j=0}^{m-1}\left[\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{[q(\zeta)]^k}\,\frac{d\zeta}{(\zeta - z_1)\cdots(\zeta - z_{j+1})}\right](z - z_1)\cdots(z - z_j) \equiv Q_k(z),\end{aligned} \tag{2.87}$$

[15] If $j = 0$, we set $(z - z_1)\cdots(z - z_j)$ equal to zero.

where, despite its subscript, $Q_k(z)$ is a polynomial of degree $m - 1$, whose coefficients are integrals of the form

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{[q(\zeta)]^k}\frac{d\zeta}{(\zeta - z_1)\cdots(\zeta - z_{j+1})}.$$

It should be noted that by using the residue theorem, $Q_k(z)$ can be expressed in terms of the values of the function $f(z)$ and its derivatives at the points $z_1, \ldots, z_m$ (the cases $m = 1, 2$ will be considered below). Comparing (2.86) and (2.87), we find that

$$\mathscr{J}_{mp-1}(z) = \sum_{k=0}^{p-1} Q_k(z)[q(z)]^k,$$

an expression called *Jacobi's interpolation polynomial* (of degree $mp - 1$).

We now examine the remainder term

$$R_{mp}(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}\frac{[q(z)]^p}{[q(\zeta)]^p}\,d\zeta, \tag{2.88}$$

corresponding to the interpolation formula $f(z) = \mathscr{J}_{mp-1}(z) + R_{mp}(z)$. To estimate (2.88), we note that the only requirements imposed on the closed rectifiable Jordan curve L is that it belong to the domain of analyticity of $f(z)$, and that its interior contain all the points $z_1, \ldots, z_m$. Consider the lemniscate of radius ρ with foci at the points $z_1, \ldots, z_m$, i.e., the curve Λ_ρ with equation

$$|q(z)| = \rho^m$$

[cf. Sec. I.83, esp. (I.17.11)]. As we know, $I(\Lambda_\rho)$ contains all the points $z_1, \ldots, z_m$, regardless of the value of $\rho > 0$. Let $\rho_0 > 0$ be such that $f(z)$ is analytic on $I(\Lambda_{\rho_0})$, and let ρ' be an arbitrary positive number less than ρ_0. Then, for the contour of integration L, we choose any lemniscate Λ_ρ corresponding to a value of ρ in the interval (ρ', ρ_0). Obviously, $I(\Lambda_{\rho_0})$ strictly includes $\overline{I(\Lambda_\rho)}$, which in turn strictly includes $\overline{I(\Lambda_{\rho'})}$.[16] Since $|q(z)| \leqslant \rho'^m$ if $z \in \overline{I(\Lambda_{\rho'})}$, while $|q(\zeta)| = \rho^m$ if $z \in \Lambda_\rho$, formula (2.88) implies

$$|R_{mp}(z)| \leqslant \frac{1}{2\pi}\frac{M(\rho)}{\delta(\rho', \rho)}\left(\frac{\rho'}{\rho}\right)^{mp} l_\rho \tag{2.89}$$

---

[16] In writing $I(\Lambda_{\rho_0})$, $\overline{I(\Lambda_\rho)}$, $\overline{I(\Lambda_{\rho'})}$, we tacitly assume that the lemniscates Λ_{ρ_0}, Λ_ρ, $\Lambda_{\rho'}$ are closed Jordan curves, i.e., have only one component each (in the sense of p. I.379). If Λ_{ρ_0} is a closed Jordan curve, we can always choose ρ and ρ' so close to ρ_0 that Λ_ρ and $\Lambda_{\rho'}$ are also closed Jordan curves. The modification of the argument if Λ_{ρ_0} is disconnected or has points of self-intersection is straightforward, and involves defining $I(\Lambda_{\rho_0})$ in the obvious way, so that $I(\Lambda_{\rho_0})$ can be the union of as many as m disjoint simply-connected domains.

if $z \in \overline{I(\Lambda_{\rho'})}$, where l_ρ is the length of Λ_ρ, $\delta(\rho', \rho)$ is the distance between $\Lambda_{\rho'}$ and Λ_ρ, and

$$M(\rho) = \max_{\zeta \in \Lambda_\rho} |f(\zeta)|.$$

It follows from (2.89) that $R_{mp}(z)$ converges uniformly to zero on $\overline{I(\Lambda_{\rho'})}$ as $p \to \infty$. Since

$$R_{mp}(z) = f(z) - \mathscr{J}_{mp-1}(z),$$

and ρ' can be chosen arbitrarily close to ρ_0, this means that

$$f(z) = \lim_{p \to \infty} \mathscr{J}_{mp-1}(z) = \lim_{p \to \infty} \sum_{k=0}^{p-1} Q_k(z)[q(z)]^k = \sum_{k=0}^{\infty} Q_k(z)[q(z)]^k, \qquad (2.90)$$

where the convergence is uniform on every compact subset of $I(\Lambda_{\rho_0})$, and the expression on the right is called *Jacobi's interpolation series*. In other words, *any function $f(z)$ analytic on $I(\Lambda_{\rho_0})$ can be expanded in a Jacobi series* (2.90) *which is uniformly convergent on every compact subset of $I(\Lambda_{\rho_0})$*.

In the case $m = 1$, recalling (2.84) and (2.87), we find that $q(z) = z - z_1$ and

$$Q_k(z) = \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^{k+1}} d\zeta = \frac{f^{(k)}(z_1)}{k!},$$

i.e., Jacobi's series reduces to Taylor's series, and the lemniscate reduces to the circle $|z - z_1| = \rho_0$. In the case $m = 2$, we have $q(z) = (z - z_1)(z - z_2)$ and

$$\begin{aligned} Q_k(z) = {} & \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^{k+1}(\zeta - z_2)^k} d\zeta \\ & + \left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^{k+1}(\zeta - z_2)^{k+1}} d\zeta\right](z - z_1). \end{aligned} \qquad (2.91)$$

If we write

$$z - z_1 = \left(z - \frac{z_1 + z_2}{2}\right) + \frac{\zeta - z_1}{2} - \frac{\zeta - z_2}{2},$$

the second term in (2.91) becomes

$$\begin{aligned} & \left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^{k+1}(\zeta - z_2)^{k+1}} d\zeta\right]\left(z - \frac{z_1 + z_2}{2}\right) \\ & + \frac{1}{2}\left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^k(\zeta - z_2)^{k+1}} d\zeta - \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^{k+1}(\zeta - z_2)^k} d\zeta\right], \end{aligned}$$

and hence

$$\begin{aligned} Q_k(z) = {} & \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{2}\left[\frac{1}{(\zeta - z_1)^k(\zeta - z_2)^{k+1}} + \frac{1}{(\zeta - z_1)^{k+1}(\zeta - z_2)^k}\right] d\zeta \\ & + \left[\frac{1}{2\pi i} \int_L \frac{f(\zeta)}{(\zeta - z_1)^{k+1}(\zeta - z_2)^{k+1}} d\zeta\right]\left(z - \frac{z_1 + z_2}{2}\right), \end{aligned} \qquad (2.92)$$

an expression which is now symmetric with respect to z_1 and z_2. Applying the residue theorem, and making some simple calculations based on (2.7), we find that

$$Q_0(z) = \frac{f(z_1) + f(z_2)}{2} + \left(z - \frac{z_1 + z_2}{2}\right) \frac{f(z_2) - f(z_1)}{z_2 - z_1},$$

while if $k \geqslant 1$,

$$\begin{aligned} Q_k(z) = \frac{1}{k!} \sum_{j=0}^{k-1} & \frac{k(k^2 - 1)\cdots[k^2 - (j-1)^2](k-j)}{j!} \\ & \times \frac{(-1)^j f^{(k-j)}(z_1) + (-1)^k f^{(k-j)}(z_2)}{2(z_1 - z_2)^{k+j}} \\ & + \frac{1}{k!}\left(z - \frac{z_1 + z_2}{2}\right) \sum_{j=0}^{k} \frac{k(k^2 - 1)\cdots[k^2 - (j-1)^2](k+j)}{j!} \\ & \times \frac{(-1)^j f^{(k-j)}(z_1) - (-1)^k f^{(k-j)}(z_2)}{(z_1 - z_2)^{k+j+1}}, \end{aligned}$$

where

$$\frac{k(k^2 - 1)\cdots[k^2 - (j-1)^2](k \pm j)}{j!}$$

is regarded as equal to 1 when $j = 0$. In particular, setting $z_1 = -z_2 = a$, we find that any function analytic inside the lemniscate

$$\Lambda(\rho_0): |(z - a)(z + a)| = \rho_0^2$$

can be expressed in a series

$$\begin{aligned} f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \Bigg\{ & \sum_{j=0}^{k-1} \frac{k(k^2 - 1)\cdots[k^2 - (j-1)^2](k-j)}{j!} \\ & \times \frac{(-1)^j f^{(k-j)}(a) + (-1)^k f^{(k-j)}(-a)}{2^{k+j+1} a^{k+j}} \\ & + \frac{z}{a} \sum_{j=0}^{k} \frac{k(k^2 - 1)\cdots[k^2 - (j-1)^2](k+j)}{j!} \\ & \times \frac{(-1)^j f^{(k-j)}(a) - (-1)^k f^{(k-j)}(-a)}{2^{k+j+1} a^{k+j}} \Bigg\} (z^2 - a^2)^k. \end{aligned} \tag{2.93}$$

It should be noted that when $I(\Lambda_{\rho_0})$ is disconnected,[17] the analytic function $f(z)$ can be defined on each component of $I(\Lambda_{\rho_0})$ independently, regardless of how $f(z)$ is defined on the other components. For example, in the case where $m = 2$ and $\rho_0 \leqslant |a|$, the lemniscate

$$|(z - a)(z + a)| = \rho_0^2$$

[17] Concerning the meaning of $I(\Lambda_{\rho_0})$ when Λ_{ρ_0} is not a closed Jordan curve, see footnote 16. As shown in Sec. I.83, $I(\Lambda_{\rho_0})$ is disconnected for all sufficiently small ρ_0.

is the boundary of two disjoint simply connected domains D_{-a} and D_a, where $-a \in D_{-a}$ and $a \in D_a$. Let A and B be constants, and suppose that

$$f(z) = \begin{cases} A & \text{if } z \in D_{-a}, \\ B & \text{if } z \in D_a. \end{cases} \tag{2.94}$$

Then (2.93) reduces to the series

$$f(z) = \frac{B + A}{2} + \frac{B - A}{2} z + \frac{B - A}{2a} \sum_{k=1}^{\infty} (-1)^k \frac{(k+1)\cdots(2k-1)2k}{k!} \frac{z(z^2 - a^2)^k}{2^{2k}a^{2k}}, \tag{2.95}$$

which converges to A on the domain D_{-a}, and to B on the domain D_a.

We can also easily derive (2.95) by starting from the binomial series

$$(1 + t)^{-1/2} = 1 + \sum_{k=1}^{\infty} \frac{(k+1)\cdots(2k-1)2k}{k!\,2^{2k}} t^k \qquad (|t| < 1), \tag{2.96}$$

and making the substitution

$$t = \frac{z^2 - a^2}{a^2}.$$

Since

$$|t| = \frac{|z^2 - a^2|}{|a|^2} < 1$$

inside the lemniscate $|z^2 - a^2| = |a|^2$, formula (2.96) implies

$$\pm\frac{a}{z} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(k+1)\cdots(2k-1)2k}{k!\,2^{2k}} \frac{(z^2 - a^2)^k}{a^{2k}}, \tag{2.97}$$

where it is clear that the plus sign should be chosen if $z = a$, and the minus sign if $z = -a$, since in both cases the right-hand side must equal 1. Therefore the plus sign should be chosen in (2.97) if z belongs to the component of $I(\Lambda_{\rho_0})$ containing the point a (one half of the "figure eight"), while the minus sign should be chosen if z belongs to the component of $I(\Lambda_{\rho_0})$ containing the point $-a$ (the other half of the "figure eight"). Multiplying (2.97) by $\frac{1}{2}(B - A)z$, we obtain

$$\pm\frac{B - A}{2} = \frac{B - A}{2a} z + \frac{B - A}{2a} \sum_{k=1}^{\infty} (-1)^k \frac{(k+1)\cdots(2k-1)2k}{k!\,2^{2k}} \frac{z(z^2 - a^2)^k}{a^{2k}},$$

which is equivalent to (2.95), where $f(z)$ is given by (2.94).

PROBLEMS

2.1. Find the residues of $f(z)$ at all its isolated singular points and at infinity (if infinity is not a limit point of singular points), where $f(z)$ is given by

a) $\dfrac{1}{z^3 - z^5}$; b) $\dfrac{1}{z(1 - z^2)}$; c) $\dfrac{z^{2n}}{(1 + z)^n}$ (n a positive integer);

d) $\dfrac{z^2}{(z^2 + 1)^2}$; e) $\dfrac{\sin 2z}{(z + 1)^3}$; f) $\dfrac{e^z}{z^2(z^2 + 9)}$;

g) $\cot^2 z$; h) $\cot^3 z$; i) $\cos \dfrac{1}{z - 2}$;

j) $z^3 \cos \dfrac{1}{z - 2}$; k) $e^{z + (1/z)}$; l) $\sin z \sin \dfrac{1}{z}$.

Ans.

a) $\operatorname*{Res}_{z = \pm 1} f(z) = -\frac{1}{2}$, $\operatorname*{Res}_{z = 0} f(z) = 1$, $\operatorname*{Res}_{z = \infty} f(z) = 0$;

d) $\operatorname*{Res}_{z = i} f(z) = -\dfrac{i}{4}$, $\operatorname*{Res}_{z = -i} f(z) = \dfrac{i}{4}$, $\operatorname*{Res}_{z = \infty} f(z) = 0$;

g) $\operatorname*{Res}_{z = k\pi} f(z) = 0$ $(k = 0, \pm 1, \pm 2, \ldots)$;

j) $\operatorname*{Res}_{z = 2} f(z) = -\operatorname*{Res}_{z = \infty} f(z) = -\frac{143}{24}$.

2.2. Find the residues of all the single-valued branches of $f(z)$ at the indicated point, where $f(z)$ is the multiple-valued function defined by

a) $\dfrac{\sqrt{z}}{1 - z}$ at $z = 1$; b) $\dfrac{1}{\sqrt{2 - z} + 1}$ at $z = 1$;

c) $\sqrt{(z - a)(z - b)}$ at $z = \infty$; d) $\dfrac{z^a}{1 - \sqrt{z}}$ at $z = 1$;

e) $\operatorname{Ln} \dfrac{z - a}{z - b}$ at $z = \infty$; f) $e^z \operatorname{Ln} \dfrac{z - a}{z - b}$ at $z = \infty$.

Ans. a) ± 1; c) $\pm \dfrac{(a - b)^2}{8}$; e) $a - b$ for all branches.

2.3. Calculate

$$\operatorname*{Res}_{z = \infty} f^2(z),$$

given that $f(z)$ has a Laurent expansion of the form

$$a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

at infinity.

2.4. Calculate

$$\operatorname*{Res}_{z = a} f(z)g(z),$$

given that $f(z)$ is analytic at $z = a$, while

a) $g(z)$ has a simple pole with residue A at $z = a$;

b) $g(z)$ has a pole of order k with principal part

$$\frac{a_{-1}}{z - a} + \cdots + \frac{a_{-k}}{(z - a)^k}$$

at $z = a$.

2.5. Calculate

$$\operatorname*{Res}_{z=a} \varphi[f(z)],$$

given that

a) $f(z)$ is analytic at $z = a$, with $f'(a) \neq 0$, while $\varphi(\zeta)$ has a simple pole with residue A at the point $\zeta = f(a)$;
b) $f(z)$ has a simple pole with residue A at $z = a$, while $\varphi(\zeta)$ has a simple pole at infinity with principal part $B\zeta$.

2.6. Prove that formula (2.12) remains valid even if some of the points (2.10) are not poles.

Hint. Show that the points which are not poles make no contribution to the sum in (2.11).

2.7. Avoiding excessive calculation, evaluate the integral

$$\int_L \frac{dz}{(z-3)(z^5-1)},$$

where L is the circle $|z| = 2$.

Hint. Use Theorem 2.6.

2.8. Use the calculus of residues to evaluate the following integrals:

a) $\int_0^{2\pi} \frac{d\varphi}{a + \cos\varphi}$ $(a > 1)$; b) $\int_0^{2\pi} \frac{d\varphi}{(a + b\cos\varphi)^2}$ $(a > b > 0)$;

c) $\int_0^{2\pi} \frac{d\varphi}{(a + b\cos^2\varphi)^2}$ $(a > 0, b > 0)$;

d) $\int_0^{2\pi} \frac{d\varphi}{1 - 2a\cos\varphi + a^2}$ (a complex, $a \neq \pm 1$).

2.9. Generalize Example 2, p. 45 to the case where $f(z)$ can have a finite number of simple poles on the real axis.

Ans.

$$\int_{-\infty}^{\infty} f(x)e^{i\lambda x}\,dx = 2\pi i \sum \operatorname{Res}\{f(z)e^{i\lambda z}\} + \pi i \sum \operatorname{Res}\{f(z)e^{i\lambda z}\}, \tag{2.98}$$

where the first sum is over all the poles of $f(z)$ in the upper half-plane, and the second sum is over all the poles of $f(z)$ lying on the real axis.

Comment. By the integral in (2.98) is meant the *Cauchy principal value* with respect to ∞ and all the poles on the real axis (cf. Problem I.13.10 and p. I.316). For example, if there is just one pole at the real point $z = a$, the integral equals

$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left[\int_{-R}^{a-\varepsilon} f(x)e^{i\lambda x}\,dx + \int_{a+\varepsilon}^{R} f(x)e^{i\lambda x}\,dx\right].$$

2.10. Evaluate the following integrals:

a) $\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 5x + 6}\, dx$; b) $\int_{-\infty}^{\infty} \frac{\sin x}{(x^2 + 4)(x - 1)}\, dx$;

c) $\int_{-\infty}^{\infty} \frac{\cos ax}{1 + x^4}\, dx \qquad (a \geqslant 0)$.

Hint. See the preceding problem.

Ans. b) $\frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$.

2.11. Let $f(z)$ be a rational function with no poles on the positive real axis, and suppose that

$$\lim_{z \to 0} z^p f(z) = \lim_{z \to \infty} z^p f(z) = 0.$$

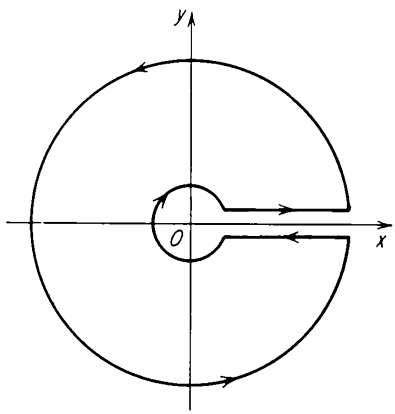

FIGURE 2.3

Prove that

$$\int_0^{\infty} x^{p-1} f(x)\, dx = \frac{\pi}{\sin p\pi} \sum \text{Res}\{(-z)^{p-1} f(z)\},$$

where the sum is over all the poles of $f(z)$.

Hint. Consider the integral

$$\int_L (-z)^{p-1} f(z)\, dz = \int_L e^{(p-1)\ln(-z)} f(z)\, dz,$$

where L is the contour shown in Figure 2.3.

2.12. Evaluate the following integrals:

a) $\int_0^{\infty} \frac{dx}{x^p(x + 1)} \qquad (0 < p < 1)$; b) $\int_0^{\infty} \frac{x^p}{1 + x^2}\, dx \qquad (-1 < p < 1)$;

c) $\int_0^{\infty} \frac{x^p}{(1 + x^2)^2}\, dx \qquad (-1 < p < 3)$.

Hint. See the preceding problem.

Ans. b) $\frac{\pi}{2} \sec \frac{p\pi}{2}$.

2.13. Prove that

$$\int_0^{\infty} \frac{\ln x}{(x^2 + 1)^2}\, dx = -\frac{\pi}{4}.$$

Hint. Consider the integral

$$\int_L \frac{\ln z}{(z^2 + 1)^2}\, dz,$$

where L is the contour shown in Figure 2.4.

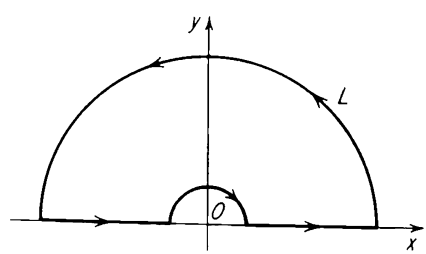

FIGURE 2.4

2.14. Evaluate the following integrals, where L is any circle with center at the origin:

a) $\frac{1}{2\pi i}\int_L \sin\frac{1}{z}\,dz$; b) $\frac{1}{2\pi i}\int_L \sin^2\frac{1}{z}\,dz$;

c) $\frac{1}{2\pi i}\int_L z^n e^{2/z}\,dz$ (n an arbitrary integer).

2.15. Use Rouché's theorem to prove the fundamental theorem of algebra (see Theorem I.17.7 and the subsequent remark).

2.16. Find the number of zeros of each of the following polynomials inside the unit circle $|z| = 1$;

a) $z^7 - 5z^4 + z^2 - 2$; b) $z^8 - 5z^5 - 2z + 1$;
c) $z^9 - 2z^6 + z^2 - 8z - 2$; d) $2z^5 - z^3 + 3z^2 - z + 8$.

2.17. How many roots has the equation

a) $z^4 - 5z + 1 = 0$ in the annulus $1 < |z| < 2$;
b) $z^4 - 8z + 10 = 0$ in the annulus $1 < |z| < 3$?

2.18. How many roots does the equation $e^z = az^n$ have inside the circle $|z| = R$, given that $|a| > e^R/R^n$?

2.19. Prove that the equation

$$z + e^{-z} = \lambda \qquad (\lambda > 1)$$

has a unique real root in the right half-plane.

2.20. Prove that, given any $\rho > 0$, the function

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots + \frac{1}{n!\,z^n}$$

has all its zeros inside the circle $|z| = \rho$ if n is sufficiently large.

Hint. Use Hurwitz's theorem.

2.21. Prove that, given any $\rho < 1$, the polynomial

$$P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$$

has no zeros inside the circle $|z| = \rho$ if n is sufficiently large.

2.22. Verify the expansion

$$\frac{1}{\sin^2 z} = \sum_{k=-\infty}^{\infty} \frac{1}{(z - k\pi)^2}.$$

2.23. Derive partial fraction expansions of the functions

a) $\tanh z$; b) $\operatorname{csch} z$.

2.24. Verify the expansion

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{2z^2}{z^2 + 4k^2\pi^2}.$$

2.25. Let $f(z)$ be a function whose only singular points in the finite plane are poles at the points $a_1, a_2, \ldots, a_m$, none of which equals $0, \pm 1, \pm 2, \ldots$. Prove that if there is a sequence of contours L_n with the same properties as in Theorem 2.7 such that

$$\lim_{n \to \infty} \int_{L_n} f(z) \cot \pi z \, dz = 0, \tag{2.99}$$

then

$$\sum_{n=-\infty}^{\infty} f(n) = -\pi \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} \{f(z) \cot \pi z\}. \tag{2.100}$$

2.26. Let $f(z)$, the points a_k and the contours L_n be the same as in the preceding problem, but suppose (2.99) is replaced by the condition

$$\lim_{n \to \infty} \int_{L_n} f(z) \csc z \, dz = 0.$$

Prove that (2.100) is then replaced by

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\pi \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} \{f(z) \csc z\}.$$

2.27. Sum the following series, where in the first four cases a is such that none of the denominators vanishes:

a) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2}$; b) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2}$; c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2+a^2}$;

d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+a^2}$; e) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$; f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

Ans. c) $\dfrac{1}{2a^2}(1 + \pi a \coth \pi a)$; f) $\dfrac{\pi^3}{32}$.

2.28. Find the polynomial of least degree which takes the same values at the points $-2, -1, 0, 1, 2$ as the function

$$f(z) = \sin \frac{\pi z}{2},$$

Ans. $\dfrac{4z - z^3}{3}$.

Comment. Thus the polynomial (2.72) may actually be of degree less than $n - 1$.

2.29. Find the polynomial of least degree which takes the same values at the points $-2, -4/3, 0, 4/3, 2$ as the function

$$f(z) = \cos \frac{\pi z}{4}.$$

2.30. Prove the following theorem: Given a set of interpolation points

$$z_1^{(n)}, \ldots, z_n^{(n)} \qquad (n = 1, 2, \ldots), \tag{2.101}$$

where in general $z_k^{(n)}$ depends on n, let

$$\omega_n(z) = (z - z_1^{(n)}) \cdots (z - z_n^{(n)}).$$

Let the distribution of the points (2.101) be such that the limit

$$U(z) = \lim_{n \to \infty} \sqrt[n]{|\omega_n(z)|} \geqslant 0$$

exists for every finite z, and moreover suppose the convergence is uniform on every bounded domain. Suppose the points (2.101) all lie inside a closed rectifiable Jordan curve L, and suppose $U(z) \geqslant \beta > 0$ on L. Let $\overline{G}$ be a closed domain contained in $I(L)$, and suppose $U(z) \leqslant \alpha$ ($\alpha > 0$) on $\overline{G}$. Then if $f(z)$ is analytic on $\overline{I(L)}$ and $\alpha < \beta$, the remainder term (2.73) in Newton's series converges uniformly to zero on $\overline{G}$.

Hint.

$$|R_n(z)| \leqslant \frac{Ml}{2\pi\delta} \frac{\max\limits_{z \in \overline{G}} |\omega_n(z)|}{\min\limits_{\zeta \in L} |\omega_n(\zeta)|},$$

where l is the length of L, δ is the distance between $\overline{G}$ and L, and

$$M = \max_{\zeta \in L} |f(\zeta)|.$$

2.31. Suppose the interpolation points (2.101) are the first n points of a sequence of distinct points converging to a finite limit a (in this case, $z_k^{(n)}$ does not depend on n). Show that then the convergence behavior of Newton's series is just the same as that of Taylor's series at the point a, i.e., Newton's series converges to $f(z)$ on the largest disk $|z - a| < \Delta$ in the domain of analyticity of $f(z)$.

Hint. $U(z) = |z - a|$.

2.32. Examine the convergence of Newton's series in the case where the interpolation points (2.101) are the distinct nth roots of unity, i.e.,

$$z_k^{(n)} = e^{2k\pi i/n} \qquad (k = 1, 2, \ldots, n).$$

Hint.

$$U(z) = \begin{cases} 1 & \text{if } |z| \leqslant 1, \\ |z| & \text{if } |z| \geqslant 1. \end{cases}$$

2.33. Construct Jacobi's interpolation polynomial for the case where the interpolation points $z_1, \ldots, z_m$ are the distinct mth roots of unity.

3

INVERSE AND IMPLICIT FUNCTIONS

12. Inverse Functions: The Single-Valued Case

Given a function $w = f(z)$, analytic at a point z_0, we now consider the problem of finding the inverse function $z = f^{-1}(w)$ at the point $w_0 = f(z_0)$. First we consider the case where $f^{-1}(w)$ is single-valued, deferring the more general multiple-valued case to the next section.

THEOREM 3.1. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a simple w_0-point. Then there is a neighborhood $\mathcal{N}(z_0)$ and a corresponding neighborhood $\mathcal{N}(w_0)$ such that every point in $\mathcal{N}(w_0)$ has a unique inverse image in $\mathcal{N}(z_0)$.*

Proof. Let $\gamma: |z - z_0| = \rho$ be a circle such that $f(z)$ has no w_0-points in $\overline{I(\gamma)}$ except at the point z_0 itself. Such a circle must exist, since otherwise z_0 would be a limit point of w_0-points, and then, by the interior uniqueness theorem,[1] $f(z)$ would be identically equal to w_0, contrary to the hypothesis that z_0 is a *simple* w_0-point [so that $f'(z_0) \neq 0$]. Choose the neighborhood $\mathcal{N}(z_0)$ to be $I(\gamma)$. Let $\Gamma = f(\gamma)$ be the image of the circle γ under the mapping $w = f(z)$, let δ be the distance between w_0 and Γ, and choose $\mathcal{N}(w_0)$ to be the disk $|w - w_0| < \delta$. Then, if w is an arbitrary point of $\mathcal{N}(w_0)$,

$$|f(z) - w_0| \geqslant \delta > |w - w_0|$$

[1] See Theorem I.17.1 or its special case, Theorem I.16.6.

for all $z \in \gamma$, and hence, by Rouché's theorem, the two functions $f(z) - w_0$ and

$$f(z) - w = f(z) - w_0 + (w_0 - w)$$

have the same number of zeros inside γ. But $f(z) - w_0$ has only one zero inside γ, by construction, and therefore the same is true of $f(z) - w$, i.e., w has a unique inverse image in $\mathcal{N}(z_0)$.

COROLLARY. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a simple w_0-point. Then the inverse function $f^{-1}(w)$ exists and is single-valued in a neighborhood of the point $w = w_0$.*

Remark. Of course, in general, $\mathcal{N}(z_0)$ contains points whose images do not lie in $\mathcal{N}(w_0)$.

THEOREM 3.2. *With the same notation as in Theorem* 3.1 *and its proof, let $g(z)$ be any function analytic on $\overline{I(\gamma)}$. Then $g[f^{-1}(w)]$ is analytic at w_0 and has the Taylor series expansion*

$$g[f^{-1}(w)] = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{[f(z) - w_0]^{n+1}}\,dz\right](w - w_0)^n \tag{3.1}$$

valid for all $w \in \mathcal{N}(w_0)$. In particular, if $g(z) \equiv z$,

$$z = f^{-1}(w) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{[f(z) - w_0]^{n+1}}\,dz\right](w - w_0)^n \tag{3.2}$$

for all $w \in \mathcal{N}(w_0)$.

Proof. According to Theorem 3.1, if $w \in \mathcal{N}(w_0)$, then $f(z)$ has a unique w-point inside γ, in fact, a simple w-point at the point $z = f^{-1}(w)$. Therefore, according to Theorem 2.2,

$$g[f^{-1}(w)] = \frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z) - w}\,dz. \tag{3.3}$$

We now apply the same transformation to this integral as used to deduce Cauchy's theorem on the expansion of an analytic function in a power series (see Theorem I.16.7) from Cauchy's integral. Since $|w - w_0| < \delta$, while $|f(z) - w| \geqslant \delta$ if $z \in \gamma$, we have

$$\begin{aligned}\frac{f'(z)}{f(z) - w} &= \frac{f'(z)}{f(z) - w_0 - (w - w_0)} \\ &= \frac{f'(z)}{f(z) - w_0}\,\frac{1}{1 - \dfrac{w - w_0}{f(z) - w_0}} = \sum_{n=0}^{\infty}\frac{f'(z)}{f(z) - w_0}\left[\frac{w - w_0}{f(z) - w_0}\right]^n.\end{aligned} \tag{3.4}$$

Writing

$$M = \max_{z \in \gamma} |f'(z)|,$$

we find that on the circle γ, the general term of the series on the right in (3.4) does not exceed

$$\frac{M}{\delta}\left(\frac{|w - w_0|}{\delta}\right)^n$$

in absolute value, and hence the series is uniformly convergent on γ. We can therefore substitute (3.4) into (3.3) and then integrate term by term, obtaining the expansion

$$\begin{aligned} g[f^{-1}(w)] &= \frac{1}{2\pi i}\int_\gamma g(z)\sum_{n=0}^{\infty}\frac{f'(z)}{f(z)-w_0}\left[\frac{w-w_0}{f(z)-w_0}\right]^n dz \\ &= \sum_{n=0}^{\infty}\left\{\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{[f(z)-w_0]^{n+1}}\,dz\right\}(w-w_0)^n, \end{aligned}$$

which is valid whenever $|w - w_0| < \delta$, i.e., for all $w \in \mathcal{N}(w_0)$. This is just the desired result (3.1).

THEOREM 3.3. *The series* (3.1) *and* (3.2) *can be written in the forms*

$$g[f^{-1}(w)] = g(z_0) + \sum_{n=1}^{\infty}\frac{1}{n!}\left\{\frac{d^{n-1}}{dz^{n-1}}g'(z)[\psi(z)]^n\right\}_{z=z_0}(w-w_0)^n \quad (3.5)$$

and

$$z = f^{-1}(w) = z_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\left\{\frac{d^{n-1}}{dz^{n-1}}[\psi(z)]^n\right\}_{z=z_0}(w-w_0)^n, \quad (3.6)$$

respectively, where

$$\psi(z) = \frac{z - z_0}{f(z) - w_0}. \quad (3.7)$$

Proof. We want to evaluate the integral

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{[f(z)-w_0]^{n+1}}\,dz,$$

figuring in the expansion (3.1). If $n = 0$, then, according to Theorem 2.2,

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)-w_0}\,dz = g(z_0). \quad (3.8)$$

If $n \geqslant 1$, we apply integration by parts,[2] obtaining

$$\begin{aligned} \frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{[f(z)-w_0]^{n+1}}\,dz &= -\frac{1}{2\pi i n}\int_\gamma g(z)\,d\left\{\frac{1}{[f(z)-w_0]^n}\right\} \\ &= \frac{1}{2\pi i n}\int_\gamma \frac{g'(z)}{[f(z)-w_0]^n}\,dz. \end{aligned} \quad (3.9)$$

[2] The usual rule for integration by parts applies to analytic functions (cf. Problem I.13.19), where the integrated term vanishes on any closed curve.

Inside γ, the function

$$\frac{g'(z)}{[f(z) - w_0]^n}$$

obviously has a pole of order n at the point z_0, but no other singular points. It follows from formula (2.7) that

$$\operatorname*{Res}_{z=z_0} \frac{g'(z)}{[f(z) - w_0]^n} = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \frac{(z - z_0)^n g'(z)}{[f(z) - w_0]^n} \right\}_{z=z_0},$$

so that (3.8) becomes

$$\frac{1}{2\pi i} \int_\gamma g(z) \frac{f'(z)}{[f(z) - w_0]^{n+1}} \, dz = \frac{1}{n!} \left\{ \frac{d^{n-1}}{dz^{n-1}} [g'(z)\psi(z)] \right\}_{z=z_0}, \quad (3.10)$$

where $\psi(z)$ is given by (3.7). Finally, substituting (3.8) and (3.10) into (3.1), we obtain (3.5), and in particular (3.6).

Remark. If $f(z)$ is analytic at the point z_0, where it has a simple w_0-point, then $f(z)$ has a Taylor series expansion

$$w = f(z) = w_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n + \cdots \quad (3.11)$$

at the point z_0, where

$$a_1 = f'(z_0) \neq 0.$$

Thus formulas (3.2) and (3.6) solve the problem of "inverting" the Taylor series (3.11). The expansion (3.6), and its generalization (3.5), are called *Lagrange's series.*

13. Inverse Functions: The Multiple-Valued Case

We now turn to the case where z_0 is a *multiple* w_0-point of the function $w = f(z)$:

THEOREM 3.4. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a w_0-point of order $k \geq 1$. Then there is a neighborhood $\mathcal{N}(z_0)$ and a corresponding neighborhood $\mathcal{N}(w_0)$ such that every point in $\mathcal{N}(w_0)$ has at least 1 and at most k distinct inverse images in $\mathcal{N}(z_0)$.*

Proof. As in the proof of Theorem 3.1, to which the present theorem reduces for $k = 1$, let $\gamma: |z - z_0| = \rho$ be a circle such that $f(z)$ has no w_0-points in $\overline{I(\gamma)}$ except at the point z_0 itself,[3] and choose the neighborhood $\mathcal{N}(z_0)$ to be $I(\gamma)$. Let $\Gamma = f(\gamma)$ be the image of the circle γ under the mapping $w = f(z)$, let δ be the distance between w_0 and Γ, and

[3] Note that now $f^{(k)}(z_0) \neq 0$.

choose $\mathcal{N}(w_0)$ to be the disk $|w - w_0| < \delta$. Then if w is an arbitrary point of $\mathcal{N}(w_0)$,

$$|f(z) - w_0| \geqslant \delta > |w - w_0|$$

for all $z \in \gamma$, and hence, by Rouché's theorem, the two functions $f(z) - w_0$ and

$$f(z) - w = f(z) - w_0 + (w_0 - w)$$

have the same number of zeros inside γ. Since $f(z) - w_0$ has k zeros inside γ,[4] by construction, the same is true of $f(z) - w$. However, $f(z)$ may have multiple w-points at points of $\mathcal{N}(z_0)$ other than z_0 itself [at every such point, the derivative $f'(z)$ vanishes, as it does at z_0]. Thus we can only assert that every $w \in \mathcal{N}(w_0)$ has at least 1 but no more than k distinct inverse images in $\mathcal{N}(z_0)$.

COROLLARY 1. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a w_0-point of order $k \geqslant 1$. Then there is a neighborhood $\mathcal{N}(z_0)$ and a corresponding neighborhood $\mathcal{N}(w_0)$ such that every point in $\mathcal{N}(w_0)$ except w_0 itself has precisely k distinct inverse images in $\mathcal{N}(z_0)$.*

Proof. We need only make the circle γ figuring in the proof of Theorem 3.4 so small that *both* $f(z) - w_0$ and $f'(z)$ do not vanish at any point of $I(\gamma)$ other than the point z_0 itself.

COROLLARY 2. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a w_0-point of order $k \geqslant 1$. Then the inverse function $f^{-1}(w)$ exists and is k-valued in a neighborhood of the point $w = w_0$.*

COROLLARY 3. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a w_0-point of order $k \geqslant 1$. Then there is a neighborhood $\mathcal{N}(z_0)$ in which every value of $f(z)$ except w_0 assumed at a point of $\mathcal{N}(z_0)$ is assumed at precisely k distinct points of $\mathcal{N}(z_0)$.*

Proof. Let $\tilde{\mathcal{N}}(z_0)$ and $\mathcal{N}(w_0)$ be the neighborhoods denoted by $\mathcal{N}(z_0)$ and $\mathcal{N}(w_0)$ in Corollary 1. There may still be points of $\tilde{\mathcal{N}}(z_0) = I(\gamma)$ which are not mapped by $w = f(z)$ into points of $\mathcal{N}(w_0)$. To prevent this, we leave $\mathcal{N}(w_0)$ unchanged, but replace the circle γ by a smaller circle $\gamma' : |z - z_0| = \rho'$ such that $f[I(\gamma')] \subset \mathcal{N}(w_0)$. Such a circle can always be found, since $f(z_0)$ is continuous at z_0. Then for the neighborhood $\mathcal{N}(z_0)$, we choose $I(\gamma')$.

COROLLARY 4. *Theorem 3.4 and its Corollaries 1–3 remain valid if $z_0 = \infty$.*

Proof. The apparatus for dealing with analyticity at ∞ and w_0-points of any order at ∞ has already been developed (see pp. I.125 and I.372),

[4] Recall that in Rouché's theorem, a zero of order k is counted as k zeros.

and involves use of the preliminary transformation $\zeta = 1/z$, together with examination of the behavior of the function $f^*(\zeta) = f(1/\zeta)$. The only new ingredient here is the observation that $z = 1/\zeta$ carries the disk $|\zeta| < \rho$ into the "disk" $|z| > 1/r$ on the Riemann sphere.

COROLLARY 5. *Theorem* 3.4 *and its Corollaries* 1–4 *remain valid if* $w_0 = \infty$, *i.e., if* $f(z)$ *has a pole of order* $k \geqslant 1$ *at* z_0.

Proof. The apparatus for reducing the study of functions with poles to that of functions with zeros has already been developed (see p. 14), and involves use of the preliminary transformation $\eta = 1/w$, together with examination of the behavior of the function $\varphi(z) = 1/f(z)$. The only new ingredient here is the observation that $w = 1/\eta$ carries the disk $|\eta| < \delta$ into the "disk" $|w| > 1/\delta$ on the Riemann sphere. In the case where $f(\infty) = \infty$, we have to consider both transformations $\zeta = 1/z$ and $\eta = 1/w$.

Although an immediate consequence of Theorem 3.4, the next result is sufficiently important to merit being called a theorem:

THEOREM 3.5. *If* $f(z) \not\equiv$ const *is analytic on a domain* G, *then* $f(G)$ *is also a domain, i.e., the image of a domain under an analytic mapping is a domain.*

Proof. The fact that $f(G)$ is connected is an immediate consequence of Theorem I.4.2. To prove that $f(G)$ is open, let w_0 be an arbitrary point of $f(G)$, and let $z_0 \in G$ be any inverse image of w_0. Clearly, z_0 is a w_0-point of order k of $f(z)$, where k is some positive integer. Therefore, by Theorem 3.4, Corollary 4, there are neighborhoods $\mathcal{N}(w_0)$ and $\mathcal{N}(z_0) \subset G$ such that every point in $\mathcal{N}(w_0)$ is the image of a point in $\mathcal{N}(z_0)$. But this implies

$$\mathcal{N}(w_0) \subset f[\mathcal{N}(z_0)] \subset f(G),$$

and hence w_0 is an interior point of $f(G)$. Thus $f(G)$ consists entirely of interior points, and the proof is complete.

COROLLARY 1. *Theorem* 3.5 *remains valid if* $f(z)$ *is allowed to have poles in* G.

Proof. Apply Theorem 3.4, Corollary 5.

COROLLARY 2 (*Maximum modulus principle*).[5] *The modulus of an analytic function* $f(z) \not\equiv$ const, *defined on a domain* G, *cannot have a maximum at any point of* G.

[5] The results in Corollaries 2 and 3 have already been encountered in Sec. I.83, where they were proved by a different method (cf. Theorem I.17.5 and its corollary).

Proof. Let z_0 be an arbitrary point of G, and let $w_0 = f(z_0)$. The image of any neighborhood of z_0 (contained in G) covers some neighborhood of w_0. Therefore, in any neighborhood of z_0 we can find a point z whose image $w = f(z)$ is further from the origin of the w-plane than the point w_0 itself, i.e., a point z_0 such that $|f(z)| > |f(z_0)|$.

COROLLARY 3 (*Minimum modulus principle*). *The modulus of an analytic function* $f(z) \not\equiv$ const, *defined on a domain* G, *cannot have a minimum at any point of* G *which is not a zero of* $f(z)$.

Proof. The proof is almost the same as that of Corollary 2, i.e., in any neighborhood of z_0 we can find a point z whose image $w = f(z)$ is closer to the origin of the w-plane than the point w_0 itself, unless w_0 is already at the origin.

Remark. Theorem I.6.1, which was proved by purely topological means, asserts that the image of a domain under a one-to-one continuous mapping is a domain. The assumption of continuity is by itself weaker than the assumption of analyticity made here, but on the other hand we do not assume that $f(z)$ is one-to-one!

We now consider the extension of Theorem 3.3 to the multiple-valued case:

THEOREM 3.6. *The generalization of Lagrange's series* (3.6) *to the case where* $f(z)$ *has a* w_0*-point of order* $k \geq 1$ *at the point* $z = z_0$ *is given by*

$$z = f^{-1}(w) = z_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \frac{d^{n-1}}{dz^{n-1}} [\psi(z)]^n \right\}_{z=z_0} (w - w_0)^{n/k}, \quad (3.12)$$

where

$$\psi(z) = \frac{z - z_0}{[f(z) - w_0]_s^{1/k}} \quad (3.13)$$

and $[f(z) - w_0]_s^{1/k}$ *denotes any single-valued branch of the* k*-valued function* $[f(z) - w_0]^{1/k}$. *Moreover,* w_0 *is an algebraic branch point of order* $k - 1$ *of the function* $z = f^{-1}(w)$.

Proof. Let $\mathcal{N}(z_0)$ and $\mathcal{N}(w_0)$ be the same as in Theorem 3.4, Corollary 1, so that every point in $\mathcal{N}(w_0)$ except w_0 has k distinct inverse images in $\mathcal{N}(z_0)$. Since $f(z)$ has a w_0-point of order k at z_0, the Taylor series expansion of $f(z)$ at z_0 is of the form

$$w = f(z) = w_0 + a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots \quad (3.14)$$

for all $z \in \mathcal{N}(z_0)$, where

$$a_k = \frac{f^{(k)}(z_0)}{k!} \neq 0.$$

Thus, writing

$$w = f(z) = w_0 + t^k,$$

we have

$$\begin{aligned} t &= [f(z) - w_0]_s^{1/k} \\ &= \sqrt[k]{a_k}(z - z_0)\left[1 + \frac{a_{k+1}}{a_k}(z - z_0) + \cdots\right]_s^{1/k}, \end{aligned} \tag{3.15}$$

where $\sqrt[k]{a_k}$ denotes any fixed kth root of a_k, and the subscript s denotes any fixed single-valued branch of the expression to which it is attached. The series

$$1 + \frac{a_{k+1}}{a_k}(z - z_0) + \cdots$$

cannot vanish at any point of $\mathcal{N}(z_0)$, since this would imply that $t = 0$ at a point of $\mathcal{N}(z_0)$ other than z_0, i.e., that $f(z) = w_0$ at a point of $\mathcal{N}(z_0)$ other than z_0, contrary to the choice of $\mathcal{N}(z_0)$. Therefore, we can actually define single-valued analytic branches of the function

$$\left[1 + \frac{a_{k+1}}{a_k}(z - z_0) + \cdots\right]^{1/k}$$

on $\mathcal{N}(z_0)$.[6] Moreover, regarded as a function of z, each branch of t has a simple zero at z_0, and no other zeros in $\mathcal{N}(z_0)$.

To find the values of z for which $f(z) = w$ $(\neq w_0)$, we argue as follows: First we find a value of t such that $w_0 + t^k$ equals w, and then we find the unique value of $z \in \mathcal{N}(z_0)$ such that $[f(z) - w_0]_s^{1/k}$ equals t. However, if t corresponds to w, so does $\sqrt[k]{1}\, t$, where $\sqrt[k]{1}$ denotes any kth root of unity. Therefore the k values of z corresponding to the k distinct numbers of the form $\sqrt[k]{1}\, t$ also correspond to w, and must be just the k distinct inverse images which w has in $\mathcal{N}(w_0)$. Thus, using Theorem 3.3, we find that

$$z = f^{-1}(w) = z_0 + \sum_{k=1}^{\infty} \frac{1}{n!}\left\{\frac{d^{n-1}}{dz^{n-1}}[\psi(z)]^n\right\}_{z=z_0} t^n, \tag{3.16}$$

where $\psi(z)$ is given by (3.13), and the effect of running through the k distinct numbers of the form $\sqrt[k]{1}\, t$ can be achieved by replacing t by the k-valued expression $(w - w_0)^{1/k}$. But then (3.16) becomes

$$z = f^{-1}(w) = z_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\left\{\frac{d^{n-1}}{dz^{n-1}}[\psi(z)]^n\right\}_{z=z_0} (w - w_0)^{n/k}.$$

---

[6] In attaching the symbol s to the function $[f(z) - w_0]^{1/k}$, where $f(z) - w_0$ vanishes at z_0 [but nowhere else in $\mathcal{N}(z_0)$], we have in mind the function

$$(z - z_0)\left[\frac{f(z) - w_0}{(z - z_0)^k}\right]_s^{1/k},$$

where the expression in brackets cannot vanish at any point of $\mathcal{N}(z_0)$.

This is just the desired result (3.12), which solves the problem of inverting the Taylor series (3.14). Obviously, (3.12) is independent of the particular single-valued branch chosen in (3.13).

Finally, if L is any Jordan curve joining the points w_0 and ∞ in the extended w-plane, then (3.12) defines k distinct single-valued analytic branches of the inverse function $z = f^{-1}(w)$ on the "cut neighborhood" $\mathcal{N}(w_0) - [\mathcal{N}(w_0) \cap L]$. Moreover, if $C \subset \mathcal{N}(w_0)$ is any closed Jordan curve such that $w_0 \in I(C)$, then one circuit around C in a fixed direction carries every branch of $f^{-1}(w)$ into another branch, while k circuits around C (*and no fewer*) carry every branch of $f^{-1}(w)$ into itself. All this is immediately apparent from the structure of (3.12) and the choice of $\mathcal{N}(w_0)$. It follows that w_0 is an algebraic branch point of order $k - 1$ of the function $f^{-1}(w)$ [see p. I.217].

14. Examples of Lagrange's Series

We now give some examples illustrating the use of the expansions (3.5) and (3.6).

Example 1. Suppose that

$$w = f(z) = ze^{-az} = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} z^{n+1} \qquad (a \neq 0),$$

and let $g(z) = e^{bz}$ $(b \neq 0)$. Then

$$z_0 = w_0 = 0, \qquad \psi(z) = \frac{z}{f(z)} = e^{az},$$

and formula (3.5) gives

$$\begin{aligned} g[f^{-1}(w)] = \exp\,[bf^{-1}(w)] &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} be^{(b+an)z}\right]_{z=0} w^n \\ &= 1 + b \sum_{n=1}^{\infty} \frac{(b+an)^{n-1}}{n!} w^k. \end{aligned} \tag{3.17}$$

Using the Cauchy-Hadamard formula [cf. (I.16.2)], we find that this series has radius of convergence[7]

$$R = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{\dfrac{|b+an|^{n-1}}{n!}}} = \frac{1}{\lim_{n\to\infty} \dfrac{e|b+an|^{1-(1/n)}}{n}} = \frac{1}{e|a|}.$$

[7] Recall that

$$\lim_{n\to\infty} \sqrt[n]{\frac{n!}{e^n}} = \frac{1}{e}$$

[cf. (I.16.6)].

If $b = a$, we have

$$g(z) = e^{az} = \frac{z}{f(z)},$$

and then (3.17) becomes

$$\frac{f^{-1}(w)}{w} = 1 + \sum_{n=1}^{\infty} a^n \frac{(n+1)^{n-1}}{n!} w^n,$$

or

$$z = f^{-1}(w) = w + \sum_{n=1}^{\infty} a^n \frac{(n+1)^{n-1}}{n!} w^{n+1} = \sum_{n=0}^{\infty} a^n \frac{(n+1)^{n-1}}{n!} w^{n+1}.$$

The same result can be obtained directly from formula (3.6).

Example 2 (*Legendre polynomials*). Next suppose that

$$w = f(z) = 2\frac{z-t}{z^2-1} \qquad (t \neq \pm 1), \tag{3.18}$$

and let $z_0 = t$. Then

$$w_0 = 0, \qquad \psi(z) = \frac{z - z_0}{f(z)} = \frac{z^2 - 1}{2},$$

and formula (3.6) gives

$$z = f^{-1}(w) = t + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} \left(\frac{z^2-1}{2}\right)^n\right]_{z=t} w^n. \tag{3.19}$$

The series (3.19) is uniformly convergent in the parameter t on every disk $|t| < R$, provided that $|w|$ is sufficiently small. In fact, writing the coefficients of the series in the form

$$\alpha_n = \frac{1}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} \left(\frac{z^2-1}{2}\right)^n\right]_{z=t} = \frac{1}{2\pi i} \int_{|z-t|=1} \frac{\left(\frac{z^2-1}{2}\right)^n}{(z-t)^n} dz,$$

we have

$$|\alpha_n| < \frac{1}{2\pi} \frac{(R+2)^{2n}}{2^n} 2\pi = \left[\frac{(R+2)^2}{2}\right]^n.$$

Therefore, if

$$|w| < \delta = \frac{2\theta}{(R+2)^2},$$

where $0 < \theta < 1$, then

$$|\alpha_n w^n| < \theta^n,$$

and the uniform convergence follows from Weierstrass' M-test (Theorem I.15.2). [Note that the convergence is also uniform in w on the disk $|w| < \delta$, provided that $|t| < R$.]

Thus it is permissible to differentiate (3.19) term by term with respect to t, for fixed w.[8] The result is

$$\frac{\partial z}{\partial t} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^n}{dz^n}\left(\frac{z^2 - 1}{2}\right)^n\right]_{z=t} w^n. \tag{3.20}$$

On the other hand, we can easily solve (3.18) for z, in terms of elementary functions:

$$z = f^{-1}(w) = \frac{1 - \sqrt{1 - 2tw + w^2}}{w}. \tag{3.21}$$

Here we choose the value of the square root which equals 1 for $w = 0$, since this choice gives $f^{-1}(0) = t$, as required. It follows from (3.21) that

$$\frac{\partial z}{\partial t} = \frac{1}{\sqrt{1 - 2tw + w^2}},$$

which, together with (3.20), gives

$$\frac{1}{\sqrt{1 - 2tw + w^2}} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left[\frac{d^n}{dz^n}(z^2 - 1)^n\right]_{z=t} w^n. \tag{3.22}$$

This series, like (3.19), is uniformly convergent in t and w for all $|t| < R$, $|w| < \delta$.

To determine the radius of convergence of (3.22) for fixed $t = t_0$, we note that if γ_ρ is the circle $|w| = \rho$, then the function in the left-hand side of (3.22) is analytic on $I(\gamma_\rho)$ for sufficiently small ρ and remains analytic on $I(\gamma_\rho)$ as ρ increases, until γ_ρ passes through a point w_0 where the square root vanishes, which occurs when

$$t_0 = \frac{1}{2}\left(w_0 + \frac{1}{w_0}\right).$$

But, as we know from Sec. I.51, the function

$$t = \frac{1}{2}\left(w + \frac{1}{w}\right) \tag{3.23}$$

transforms γ_ρ into an ellipse E_ρ, with foci ± 1 and semiaxes

$$\frac{1}{2}\left(\frac{1}{\rho} + \rho\right), \qquad \frac{1}{2}\left(\frac{1}{\rho} - \rho\right).$$

[8] Of course, we also use the fact that, for fixed w, the terms of (3.19) are analytic functions of t (cf. Theorem I.15.6).

Moreover, every point of the finite plane belongs to some ellipse E_ρ $(0 < \rho \leqslant 1)$.[9] Let E_{ρ_0} be the ellipse containing the point t_0. Then the function

$$\frac{1}{\sqrt{1 - 2tw + w^2}} \tag{3.24}$$

is analytic on the disk $I(\gamma_{\rho_0})$, and hence the series (3.22), regarded as a power series in the variable w, converges on $I(\gamma_{\rho_0})$. Since $(z^2 - 1)^n$ is a polynomial of degree $2n$ in z, the expression

$$\frac{d^n}{dz^n}(z^2 - 1)^n$$

is a polynomial of degree n in z, and hence the coefficients of the series (3.22) are polynomials of degree n in t, which we denote by $P_n(t)$ and call the *Legendre polynomials*. Thus (3.22) takes the form

$$\frac{1}{\sqrt{1 - 2tw + w^2}} = \sum_{n=0}^{\infty} P_n(t)w^n, \tag{3.25}$$

where $P_n(t)$ is given by *Rodrigues' formula*

$$P_n(t) = \frac{1}{2^n n!}\left\{\frac{d^n}{dz^n}(z^2 - 1)^n\right\}_{z=t} = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2 - 1)^n \qquad (n = 0, 1, 2, \ldots).$$

Because of the expansion (3.25), we call (3.24) the *generating function* of the Legendre polynomials.

We now derive a number of remarkable properties of the Legendre polynomials:

1. Taking the logarithmic derivative of the quantity

$$s = (t^2 - 1)^n,$$

we have

$$\frac{s'}{s} = \frac{2nt}{t^2 - 1},$$

or

$$(t^2 - 1)s' = 2nts. \tag{3.26}$$

Differentiating (3.26) $n + 1$ times we obtain

$$(t^2 - 1)s^{(n+2)} + 2(n + 1)ts^{(n+1)} + n(n + 1)s^{(n)} = 2nts^{(n+1)} + 2n(n + 1)s^{(n)}. \tag{3.27}$$

Then, since

$$P_n(t) = \frac{1}{2^n n!}s^{(n)},$$

[9] The case $\rho = 1$ is exceptional. Then the ellipse reduces to the segment of the real axis joining the points -1 and 1, traversed twice (cf. p. I.198). It should be noted that (3.23) maps $I(\gamma_\rho)$, where $0 < \rho < 1$, onto the *exterior* of E_ρ.

(3.27) reduces to the following second-order linear differential equation satisfied by $P_n(t)$:

$$(t^2 - 1)P_n''(t) + 2tP_n'(t) - n(n+1)P_n(t) = 0.$$

2. Differentiating (3.25) term by term with respect to w, we obtain

$$\frac{t - w}{(1 - 2tw + w^2)^{3/2}} = \sum_{n=1}^{\infty} nP_n(t)w^{n-1},$$

which implies

$$\begin{aligned}\frac{t - w}{\sqrt{1 - 2tw + w^2}} &= (1 - 2tw + w^2)\sum_{n=1}^{\infty} nP_n(t)w^{n-1} \\ &= P_1(t) + 2[P_2(t) - tP_1(t)]w \\ &\quad + \sum_{n=2}^{\infty} [(n+1)P_{n+1}(t) - 2ntP_n(t) + (n-1)P_{n-1}(t)]w^n.\end{aligned} \tag{3.28}$$

On the other hand,

$$\begin{aligned}\frac{t - w}{\sqrt{1 - 2tw + w^2}} &= (t - w)\sum_{n=0}^{\infty} P_n(t)w^n \\ &= t + [tP_1(t) - 1]w + \sum_{n=2}^{\infty} [tP_n(t) - P_{n-1}(t)]w^n,\end{aligned} \tag{3.29}$$

and hence, comparing coefficients of identical powers of w^n in (3.28) and (3.29), we find that

$$\begin{aligned}&P_1(t) = t, \qquad 2P_2(t) - 3tP_1(t) + 1 = 0, \\ &(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0 \qquad (n \geqslant 2).\end{aligned}$$

These relations allow us to calculate the Legendre polynomials step by step. The first few Legendre polynomials turn out to be

$$\begin{aligned}&P_0(t) = 1, \qquad P_1(t) = t, \qquad P_2(t) = \tfrac{1}{2}(3t^2 - 1), \\ &P_3(t) = \tfrac{1}{2}(5t^3 - 3t), \qquad P_4(t) = \tfrac{1}{2}(35t^4 - 30t^2 + 3), \ldots\end{aligned}$$

3. Subtracting the differential equation

$$(1 - t^2)P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0$$

multiplied by $P_m(t)$, from the differential equation

$$(1 - t^2)P_m''(t) - 2tP_m'(t) + m(m+1)P_m(t) = 0$$

multiplied by $P_n(t)$, we find that

$$\begin{aligned}&(1 - t^2)[P_m''(t)P_n(t) - P_n''(t)P_m(t)] - 2t[P_m'(t)P_n(t) - P_n'(t)P_m(t)] \\ &\quad + (m - n)(m + n + 1)P_m(t)P_n(t) = \frac{d}{dt}\{(1 - t^2)[P_m'(t)P_n(t) - P_n'(t)P_m(t)]\} \\ &\qquad + (m - n)(m + n + 1)P_m(t)P_n(t) = 0.\end{aligned} \tag{3.30}$$

Integration of (3.30) from -1 to 1 gives

$$\{(1 - t^2)[P_m'(t)P_n(t) - P_n'(t)P_m(t)]\}_{t=-1}^{t=1} + (m - n)(m + n + 1)\int_{-1}^{1} P_m(t)P_n(t) = 0.$$

Since the integrated term vanishes, we have

$$(m - n)(m + n + 1)\int_{-1}^{1} P_m(t)P_n(t)\,dt = 0,$$

and hence

$$\int_{-1}^{1} P_m(t)P_n(t)\,dt = 0 \qquad \text{if} \quad m \neq n.$$

In other words, the Legendre polynomials are *orthogonal* on the interval $[-1, 1]$.

Example 3 (*Kepler's equation*). A problem of considerable importance in astronomy, arising in the study of planetary motion, is the determination of a quantity E, called the "eccentric anomaly" of the planet, which satisfies *Kepler's equation*[10]

$$E - e \sin E = 2\pi \frac{t - T}{U},$$

where e is the eccentricity of the planet's orbit, $t - T$ is the elapsed time since the planet's last passage through perihelion, and U is the period of a complete revolution of the planet around the sun. Replacing E by z, e by w and $2\pi(t - T)/U$ by τ,[11] we can write Kepler's equation in the form

$$w = f(z) = \frac{z - \tau}{\sin z}.$$

Let $z_0 = \tau \neq k\pi$ $(k = 0, 1, \ldots)$, so that

$$w_0 = 0, \qquad \psi(z) = \frac{z - z_0}{f(z)} = \sin z.$$

Then formula (3.6) gives

$$z = \tau + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} (\sin z)^n\right]_{z=\tau} w^n. \tag{3.31}$$

This series solves the problem of determining $z = E$ from a knowledge of $w = E$ and

$$\tau = 2\pi \frac{t - T}{U}.$$

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[10] See e.g., S. W. McCuskey, *Introduction to Celestial Mechanics*, Addison-Wesley Publishing Co., Inc., Reading, Mass (1963), p. 47.

[11] By its very nature, τ must lie in the interval $0 \leqslant \tau < 2\pi$.

To find the radius of convergence of (3.31), we retrace the steps leading to the derivation of Lagrange's series (3.6). Suppose $f(z)$ is analytic at the point $z = z_0$, where it has a simple w_0-point. Let $\gamma_\rho: |z - z_0| = \rho$ be a circle such that $f(z)$ has no w_0-points in $\overline{I(\gamma_\rho)}$ except at z_0, and let $\delta(\rho) > 0$ be the distance between w_0 and $\Gamma_\rho = f(\gamma_\rho)$, i.e.,

$$\delta(\rho) = \min_{z \in \gamma_\rho} |f(z) - w_0|.$$

(For a reason that will soon be apparent, we now explicitly indicate which quantities depend on ρ.) Then the series (3.6) certainly converges on the disk $|w - w_0| < \delta(\rho)$. However, the size of the disk depends on the choice of ρ, and we want to maximize $\delta(\rho)$ for a given z_0. To illustrate how this is done in the present example, let $z_0 = \tau = \pi/2$, so that any value of ρ in the interval $(0, \pi/2)$ is admissible, since the distance between τ and the nearest pole of $f(z)$ is $\pi/2$ [$f(z)$ has poles at $z = 0$ and $z = \pi$, where $\sin z$ vanishes but no zeros except at $z = \pi/2$]. Then we have

$$\begin{aligned}\delta(\rho) &= \min_{|z - \frac{1}{2}\pi| = \rho} |f(z)| = \min_{|z - \frac{1}{2}\pi| = \rho} \left|\frac{z - \tau}{\sin z}\right| \\ &= \min_{|z - \frac{1}{2}\pi| = \rho} \frac{\rho}{|\sin z|} = \frac{\rho}{\max\limits_{|z - \frac{1}{2}\pi| = \rho} |\sin z|}.\end{aligned}$$

But

$$\begin{aligned}\max_{|z - \frac{1}{2}\pi| = \rho} |\sin z| &= \max_{0 \leqslant \varphi \leqslant 2\pi} \left|\sin\left(\frac{\pi}{2} + \rho e^{i\varphi}\right)\right| \\ &= \max_{0 \leqslant \varphi \leqslant 2\pi} |\cos(\rho \cos\varphi + i\rho \sin\varphi)| = \cosh\rho,\end{aligned}$$

where the last step follows from the fact that on the one hand,

$$|\cos(\rho \cos\varphi + i\rho \sin\varphi)| \leqslant |\cosh(\rho \sin\varphi)| \leqslant \cosh\rho$$

[cf. (I.9.48)], while on the other hand,

$$\left|\cos\left(\rho \cos\frac{\pi}{2} + i\rho \sin\frac{\pi}{2}\right)\right| = |\cos(i\rho)| = \cosh\rho.$$

Therefore

$$\delta(\rho) = \frac{\rho}{\cosh\rho},$$

and maximizing $\delta(\rho)$ on the interval $(0, \pi/2)$, we find that $\delta(\rho)$ takes its largest value for $\rho = \rho_0$, where ρ_0 satisfies the equation

$$\rho_0 = \coth\rho_0 \quad \text{or} \quad e^{2\rho_0} = \frac{\rho_0 + 1}{\rho_0 - 1},$$

whose solution turns out to be

$$\rho_0 = 1.1997\ldots < \frac{\pi}{2}.$$

The corresponding maximum value of $\delta(\rho)$ is

$$\delta(\rho_0) = \sqrt{\rho_0^2 - 1} = 0.6627\ldots.$$

Therefore, if $\tau = \pi/2$, the series (3.21) converges for $|w| < \delta(\rho_0)$. Moreover, it can be shown that $\delta(\rho_0)$ is actually the radius of convergence of (3.31) [see Problem 3.11].

*15. Functions of Two Complex Variables

The results obtained above, involving the inversion of Taylor series, can be regarded as a special case of a more general theory of *implicit functions*, to which the rest of this chapter will be devoted. We begin by sketching the theory of functions of two complex variables, confining ourselves to bare essentials.

DEFINITION 1. *By the Cartesian product of two given sets E_1 and E_2, written $E_1 \times E_2$, we mean the set of all ordered pairs (z_1, z_2), where $z_1 \in E_1$, $z_2 \in E_2$. The pairs (z_1, z_2) are called points of $E_1 \times E_2$.*

DEFINITION 2. *Let G be a domain in the z-plane and D a domain in the w-plane, and let $F(z, w)$ be a single-valued continuous function on the four-dimensional domain $G \times D$.*[12] *Then $F(z, w)$ is said to be analytic on $G \times D$ if $F(z, w)$ is analytic on G for every fixed $w \in D$ and analytic on D for every fixed $z \in G$.*

THEOREM 3.7 (*Generalization of Cauchy's integral formula*). *Given a domain G in the z-plane and a domain D in the w-plane, let $F(z, w)$ be analytic on $G \times D$. Let C and Γ be closed rectifiable Jordan curves such that $\overline{I(C)} \subset G$ and $\overline{I(\Gamma)} \subset D$. Then*

$$\begin{aligned} F(z, w) &= -\frac{1}{4\pi^2} \int_C d\zeta \int_\Gamma \frac{F(\zeta, \eta)}{(\zeta - z)(\eta - w)}\, d\eta \\ &= -\frac{1}{4\pi^2} \int_\Gamma d\eta \int_C \frac{F(\zeta, \eta)}{(\zeta - z)(\eta - w)}\, d\zeta \end{aligned} \tag{3.32}$$

for every point $(z, w) \in I(C) \times I(\Gamma)$.

[12] The set $G \times D$, which will be called a *domain*, is four-dimensional since each of its points is specified by four real numbers, the real and imaginary parts of z, and the real and imaginary parts of w. We say that $F(z, w)$ is continuous at the point (z_0, w_0) if, given any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $|F(z, w) - F(z_0, w_0)| < \varepsilon$ whenever $|z - z_0| < \delta$, $|w - w_0| < \delta$. Continuity on $G \times D$ means continuity at every point of $G \times D$. Any domain defined by a pair of inequalities $|z - z_0| < r$, $|w - w_0| < \rho$ will be called a *neighborhood* of the point (z_0, w_0).

Proof. We need only make two applications of Cauchy's integral formula (Theorem I.14.1). Thus, for example,

$$-\frac{1}{4\pi^2}\int_C \frac{d\zeta}{\zeta - z}\int_\Gamma \frac{F(\zeta,\eta)}{\eta - w}\,d\eta = \frac{1}{2\pi i}\int_C \frac{F(\zeta, w)}{\zeta - z}\,d\zeta = F(z, w),$$

and similarly for the second integral in (3.32).

Remark. Since the order of integration is unimportant, we henceforth write (3.32) as

$$F(z, w) = -\frac{1}{4\pi^2}\iint_{C\times\Gamma} \frac{F(\zeta,\eta)}{(\zeta - z)(\eta - w)}\,d\zeta\,d\eta, \tag{3.33}$$

where the right-hand side denotes either integral in (3.32).

THEOREM 3.8. *With the same hypotheses and notation as in Theorem* 3.7,

$$\frac{\partial^{k+m}F(z, w)}{\partial z^k\,\partial w^m} = -\frac{k!\,m!}{4\pi^2}\iint_{C\times\Gamma} \frac{F(\zeta,\eta)}{(\zeta - z)^{k+1}(\eta - w)^{m+1}}\,d\zeta\,d\eta \tag{3.34}$$

for every point $(z, w) \in I(C) \times I(\Gamma)$. *Moreover, every derivative* (3.34) *is analytic on* $G \times D$.

Proof. To prove that formula (3.34) holds, we need only apply Theorem I.14.5, Corollary 1 twice to formula (3.33). To prove that the function (3.34) is continuous on $G \times D$, let the circles $c: |z - z_0| = r$, $L: |z| = R_1$ be such that $\overline{I(c)} \subset I(C)$, $c \subset I(L)$, and let δ_1 be the distance between c and C. Similarly, let the circles $\gamma: |w - w_0| = \rho$, $\Lambda: |w| = R_2$ be such that $\overline{I(\gamma)} \subset I(\Gamma)$, $\Gamma \subset I(\Lambda)$, and let δ_2 be the distance between γ and Γ. Then if

$$R = \max(R_1, R_2), \qquad \delta = \min(\delta_1, \delta_2), \qquad M = \max_{\substack{\zeta\in C\\ \eta\in\Gamma}} |F(\zeta,\eta)|,$$

we have the estimate

$$\begin{aligned}
&\left|\frac{\partial^{k+m}F(z, w)}{\partial z^k\partial w^m} - \frac{\partial^{k+m}F(z_0, w_0)}{\partial z^k\partial w^m}\right| \\
&= \frac{k!m!}{4\pi^2}\left|\iint_{C\times\Gamma} F(\zeta,\eta)\,\frac{(\zeta-z_0)^{k+1}(\eta-w_0)^{m+1} - (\zeta-z)^{k+1}(\eta-w)^{m+1}}{(\zeta-z_0)^{k+1}(\eta-w_0)^{m+1}(\zeta-z)^{k+1}(\eta-w)^{m+1}}\,d\zeta\,d\eta\right| \\
&= \frac{k!m!}{4\pi^2}\left|\iint_{C\times\Gamma} F(\zeta,\eta)\right. \\
&\quad\left.\times\,\frac{(\zeta-z_0)^{k+1}[(\eta-w_0)^{m+1} - (\eta-w)^{m+1}] + (\eta-w)^{m+1}[(\zeta-z_0)^{k+1} - (\zeta-z)^{k+1}]}{(\zeta-z_0)^{k+1}(\eta-w_0)^{m+1}(\zeta-z)^{k+1}(\eta-w)^{m+1}}\,d\zeta\,d\eta\right| \\
&\leqslant \frac{k!m!}{4\pi^2}\,\frac{(2R)^{k+1}(m+1)\rho(2R)^m + (2R)^{m+1}(k+1)r(2R)^k}{\delta^{2k+2m+4}}\,Ml_Cl_\Gamma \\
&= \frac{k!m!}{4\pi^2}\left(\frac{2R}{\delta}\right)^{2k+2m+2}\frac{(m+1)\rho + (k+1)r}{\delta^2}\,Ml_Cl_\Gamma,
\end{aligned} \tag{3.35}$$

where l_C is the length of C and l_Γ the length of Γ.[13] But obviously, for sufficiently small r and ρ, the expression on the right can be made smaller than any given $\varepsilon > 0$. This proves that (3.34) is continuous on $I(C) \times I(\Gamma)$, and hence on $G \times D$, because of the arbitrariness of $C \subset G$ and $\Gamma \subset D$. Finally, (3.34) is analytic on $I(C)$ for every $w \in I(\Gamma)$ and analytic on $I(\Gamma)$ for every $z \in I(C)$, since the derivatives

$$\frac{\partial^{k+1+m} F(z, w)}{\partial z^{k+1}\, \partial w^m}, \qquad \frac{\partial^{k+m+1} F(z, w)}{\partial z^k\, \partial w^{m+1}}$$

exist, again by Theorem I.14.5, Corollary 1. This fact, together with the continuity just proved and the arbitrariness of C and Γ, shows that the function (3.34) is analytic on $G \times D$, as asserted.

THEOREM 3.9. *Given a domain G in the z-plane and a domain D in the w-plane, let $F(z, w)$ be analytic on $G \times D$. Let (z_0, w_0) be an arbitrary point of $G \times D$, and let δ be the distance from z_0 to the boundary of G and Δ the distance from w_0 to the boundary of D. Then there exists a "double power series"*

$$F(z, w) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} A_{km}(z - z_0)^k (w - w_0)^m$$

converging to $F(z, w)$ on the "four-dimensional disk" $K_1 \times K_2$, where $K_1: |z - z_0| < \delta$ and $K_2: |w - w_0| < \Delta$.

Proof. The statement of the theorem and its proof are straightforward generalizations of Cauchy's theorem on the expansion of an analytic function in a power series (Theorem I.16.7). Let z be an arbitrary point of K_1 and let c_r be any circle with center z_0 and radius r $(0 < r < \delta)$ such that $z \in I(c_r)$. Similarly, let w be an arbitrary point of K_2, and let γ_ρ be any circle with center w_0 and radius ρ $(0 < \rho < \Delta)$ such that $w \in I(\gamma_\rho)$. Then, according to (3.33), we have

$$F(z, w) = -\frac{1}{4\pi^2} \iint_{c_r \times \gamma_\rho} \frac{F(\zeta, \eta)}{(\zeta - z)(\eta - w)}\, d\zeta\, d\eta.$$

By the usual device,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}, \tag{3.36}$$

and similarly,

$$\frac{1}{\eta - w} = \sum_{m=0}^{\infty} \frac{(w - w_0)^m}{(\eta - w_0)^{m+1}}. \tag{3.37}$$

[13] In deriving (3.35), we use the formula

$$a^{n+1} - b^{n+1} = (a - b)(a^n + a^{n-1}b + \cdots + b^n).$$

Moreover, since[14]

$$\left|\frac{z - z_0}{\zeta - z_0}\right| < 1, \qquad \left|\frac{w - w_0}{\eta - w_0}\right| < 1,$$

both series (3.36) and (3.37) are absolutely convergent, and can therefore be multiplied (cf. Theorem I.3.7), with the result

$$\frac{1}{(\zeta - z)(\eta - w)} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(z - z_0)^k (w - w_0)^m}{(\zeta - z_0)^{k+1} (\eta - w_0)^{m+1}}. \tag{3.38}$$

Note that the series on the right in (3.38) is also absolutely convergent, so that its terms can be arranged in any order.

Next we multiply (3.38) by the function $-(1/4\pi^2)F(\zeta, \eta)$, obtaining

$$-\frac{1}{4\pi^2} \frac{F(\zeta, \eta)}{(\zeta - z)(\eta - w)} = -\frac{1}{4\pi^2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{F(\zeta, \eta)}{(\zeta - z_0)^{k+1} (\eta - w_0)^{m+1}} (z - z_0)^k (w - w_0)^m.$$

But the series on the right, regarded as a function of ζ and η, is uniformly convergent on $c_r \times \gamma_\rho$, by the obvious double-series generalization of Weierstrass' M-test (Theorem I.15.2), where we use a convergent "double geometric series"

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a^k b^m \qquad (0 < a < 1, 0 < b < 1)$$

as the comparison series. Therefore we can integrate (3.40) term by term "along" $c_r \times \gamma_\rho$ obtaining

$$\begin{aligned} F(z, w) &= -\frac{1}{4\pi^2} \iint_{c_r \times \gamma_\rho} \frac{F(\zeta, \eta)}{(\zeta - z)(\eta - w)} \, d\zeta \, d\eta \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} A_{km} (z - z_0)^k (w - w_0)^m, \end{aligned} \tag{3.39}$$

where

$$A_{km} = -\frac{1}{4\pi^2} \iint_{c_r \times \gamma_\rho} \frac{F(\zeta, \eta)}{(\zeta - z_0)^{k+1} (\eta - w_0)^{m+1}} \, d\zeta \, d\eta. \tag{3.40}$$

But (3.39) is just the desired result, since the points $z \in K_1$, $w \in K_2$ are arbitrary.

Remark. It follows from (3.40) and Theorem 3.8 that

$$A_{km} = \frac{1}{k!\, m!} \frac{\partial^{k+m} F(z_0, w_0)}{\partial z^k \, \partial w^m}. \tag{3.41}$$

[14] Recall that $z \in I(c_r)$, $\zeta \in c_r$ and $w \in I(\gamma_\rho)$, $\eta \in \gamma_\rho$.

On the other hand, if

$$M(r, \rho) = \max_{\substack{\zeta \in C_r \\ \eta \in \gamma_\rho}} |F(\zeta, \eta)|,$$

we find from (3.40) that

$$|A_{km}| \leqslant \frac{M(r, \rho) 2\pi r \cdot 2\pi\rho}{4\pi^2 r^{k+1} \rho^{m+1}} = \frac{M(r, \rho)}{r^k \rho^m}. \tag{3.42}$$

Together, (3.41) and (3.42) imply the following generalization of *Cauchy's inequalities* [cf. (I.14.22)]:

$$\left| \frac{\partial^{k+m} F(z_0, w_0)}{\partial z^k \, \partial w^m} \right| \leqslant k!\, m! \, \frac{M(r, \rho)}{r^k \rho^m} \qquad (k, m = 0, 1, 2, \ldots).$$

Moreover, using (3.41), we can write (3.39) as a Taylor series

$$F(z, w) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!\, m!} \frac{\partial^{k+m} F(z_0, w_0)}{\partial z^k \, \partial w^m} (z - z_0)^k (w - w_0)^m \tag{3.43}$$

in the two complex variables z and w.

*16. Weierstrass' Preparation Theorem. The Implicit Function Theorem

We now turn to the theory of implicit functions. Our first result shows that under certain conditions an implicit function $w = w(z)$, defined by an identity $F[z, w(z)] \equiv 0$, behaves locally like the solution of an algebraic equation:

THEOREM 3.10 (*Weierstrass' preparation theorem*). *Let $F(z, w)$ be a function of two complex variables which is analytic in a neighborhood $|z - z_0| < r$, $|w - w_0| < \rho$ of the point (z_0, w_0), and suppose that*

$$F(z_0, w_0) = 0, \qquad F(z_0, w) \not\equiv 0. \tag{3.44}$$

Then there is a neighborhood $|z - z_0| < r' < r$, $|w - w_0| < \rho' < \rho$ in which $F(z, w)$ can be written as

$$F(z, w) = [A_0(z) + A_1(z)w + \cdots + A_{k-1}(z)w^{k-1} + w^k]G(z, w), \tag{3.45}$$

where k is such that[15]

$$\frac{\partial F(z_0, w_0)}{\partial w} = \cdots = \frac{\partial^{k-1} F(z_0, w_0)}{\partial w^{k-1}} = 0, \qquad \frac{\partial^k F(z_0, w_0)}{\partial w^k} \neq 0,$$

[15] The existence of such an integer $k \geqslant 1$ follows at once from (3.44).

the functions $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ *are analytic if* $|z - z_0| < r'$, *and the function* $G(z, w)$ *is analytic and nonzero if* $|z - z_0| < r'$, $|w - w_0| < \rho'$.[16]

Proof. Clearly, the function $F(z_0, w)$ has a zero of order k at the point w_0. Let $\gamma' : |w - w_0| = \rho' < \rho$ be a circle such that $F(z_0, w)$ has no zeros in $\overline{I(\gamma')}$ except at the point w_0 itself. Let $\mu > 0$ be the minimum of $|F(z_0, w)|$ on the circle γ', and choose a circle $c' : |z - z_0| = r' < r$ so small that

$$|F(z, w) - F(z_0, w)| < \mu$$

for all $z \in \overline{I(c')}$ and $w \in \gamma'$. Then, according to Rouché's theorem, the two functions $F(z_0, w)$ and

$$F(z, w) = F(z_0, w) + [F(z, w) - F(z_0, w)],$$

where z is a fixed but arbitrary point of $\overline{I(c')}$, have the same number of zeros inside γ', in fact, exactly k zeros (with due regard for order), which we denote by $w_1(z), \ldots, w_k(z)$.[17] Let $P(z, w)$ be the polynomial in w which has precisely these zeros, i.e., let

$$\begin{aligned} P(z, w) &= \prod_{j=1}^{k} [w - w_j(z)] \\ &= w^k - [w_1(z) + \cdots + w_k(z)]w^{k-1} \\ &\quad + [w_1(z)w_2(z) + \cdots + w_{k-1}(z)w_k(z)]w^{k-2} \\ &\quad - \cdots + (-1)^k w_1(z) \cdots w_k(z) \\ &= w^k + A_{k-1}(z)w^{k-1} + A_{k-2}(z)w^{k-2} + \cdots + A_0(z). \end{aligned}$$

Moreover, let $s_m(z)$ denote the sum of the mth powers of all the zeros of $P(z, w)$:

$$s_m(z) = w_1^m(z) + \cdots + w_k^m(z) \qquad (m = 1, \ldots, k).$$

Then it is not hard to prove the following relation between the coefficients $A_j(z)$ and the functions $s_m(z)$ [see Problem 3.14]:

$$s_m(z) + s_{m-1}(z)A_{k-1}(z) + s_{m-2}(z)A_{k-2}(z) + \cdots + mA_{k-m}(z) = 0 \qquad (m = 1, \ldots, k). \quad (3.46)$$

According to (3.46), every coefficient $A_j(z)$ can be expressed as a polynomial in the functions $s_m(z)$, with numerical coefficients. Thus, for example,

$$\begin{aligned} A_{k-1}(z) &= -s_1(z), \\ A_{k-2}(z) &= \tfrac{1}{2}[s_1(z)]^2 - \tfrac{1}{2}s_2(z), \\ &\cdots \end{aligned}$$

[16] Thus, despite the seeming generality of the equation $F(z, w) = 0$, there is a neighborhood of the point (z_0, w_0) where it is equivalent to the equation

$$A_0(z) + A_1(z)w + \cdots + A_{k-1}(z)w^{k-1} + w^k = 0,$$

which is *algebraic* in w.

[17] Of course, $F(z, w)$ may have other zeros *outside* γ'.

Therefore, if the functions $s_1(z), \ldots, s_k(z)$ are analytic in a neighborhood of the point z_0, the same is true of the functions $A_0(z), \ldots, A_{k-1}(z)$. The functions $s_m(z)$ are easily investigated, by using Theorem 2.2, which implies

$$\frac{1}{2\pi i}\int_{\gamma'} w^m \frac{\dfrac{\partial F(z, w)}{\partial w}}{F(z, w)}\, dw = w_1^m(z) + \cdots + w_k^m(z) = s_m(z), \qquad z \in \overline{I(c')}, \tag{3.47}$$

since inside γ', $F(z, w)$ has zeros at the points $w_1(z), \ldots, w_k(z)$, and no poles at all. But $F(z, w)$ and $(\partial/\partial w)F(z, w)$ are analytic on $I(c')$ for every $w \in \gamma'$, and moreover $F(z, w)$ does not vanish on the set $I(c') \times \gamma'$, since

$$|F(z, w) - F(z_0, w)| < \mu \leqslant F(z_0, w)$$

on $\overline{I(c')} \times \gamma'$. Thus the function

$$\Phi(z, w) = w^m \frac{\dfrac{\partial F(z, w)}{\partial w}}{F(z, w)}$$

meets all the requirements of Theorem I.17.19, with L replaced by γ' and G by $I(c')$.[18] It follows that the function defined by the integral (3.47) is analytic on $I(c')$, for every $m = 1, \ldots, k$. Therefore the same is true of the functions $s_1(z), \ldots, s_k(z)$, and of the functions $A_0(z), \ldots, A_{k-1}(z)$.

We must still show that the ratio

$$G(z, w) = \frac{F(z, w)}{P(z, w)}$$

is analytic and nonzero on $I(c') \times I(\gamma')$. It is immediately apparent that $G(z, w)$ is nonzero on $I(c') \times I(\gamma')$ and analytic on $I(\gamma')$ for every $z \in I(c')$, since $F(z, w)$ and $P(z, w)$ are analytic on $I(\gamma')$ and have the same zeros $w_1(z), \ldots, w_k(z)$ inside γ', for every $z \in I(c')$. However, it is a subtler matter to prove that $G(z, w)$ is analytic on $I(c')$ for every $w \in I(\gamma')$, and that $G(z, w)$ is continuous, as a function of two variables, on $I(c') \times I(\gamma')$.

To show that $G(z, w)$ is analytic on $I(c')$ for every $w \in I(\gamma')$, we argue as follows: For every $z \in I(c')$, the zeros of $F(z, w)$ lie inside γ', i.e., $|w_j(z) - w_0| < \rho'$, and hence the function $P(z, w)$ is nonzero on $I(c') \times \gamma'$. Let r'' be any number in the interval $(0, r')$, and let c'' be the circle $|z - z_0| = r''$. Then, since the functions $A_0(z), \ldots, A_{k-1}(z)$

[18] The uniform boundedness of the family of functions $\Phi(z, w)$, $w \in \gamma'$ inside $I(c')$, i.e., on every compact subset of $I(c')$, follows from the continuity of $\Phi(z, w)$ on the closed set $\overline{I(c')} \times \gamma'$.

are analytic on $I(c')$, they are continuous on $\overline{I(c'')}$, and hence $P(z, w)$ is continuous on the closed set $\overline{I(c'')} \times \overline{I(\gamma')}$ as well as nonzero on $\overline{I(c'')} \times \gamma'$. It follows that there is a number ρ'' in the interval $(0, \rho')$ so close to ρ' that $P(z, w)$ is nonzero on the set $I(c'') \times \gamma''$, where γ'' is the circle $|w - w_0| = \rho''$. Thus we have

$$G(z, w) = \frac{1}{2\pi i} \int_{\gamma''} \frac{G(z, \eta)}{\eta - w}\, d\eta = \frac{1}{2\pi i} \int_{\gamma''} \frac{F(z, \eta)}{P(z, \eta)(\eta - w)}\, d\eta \qquad (3.48)$$

on the set $\overline{I(c'')} \times I(\gamma'')$. For fixed $w \in I(\gamma'')$, the function

$$\Phi_w(z, \eta) = \frac{F(z, \eta)}{P(z, \eta)(\eta - w)} \qquad (3.49)$$

(indexed by the parameter w) is analytic on $I(c'')$ for every $\eta \in \gamma''$, since the same is true of the numerator and the (nonvanishing) denominator. Therefore, for every $w \in I(\gamma'')$, the function $\Phi_w(z, \eta)$ meets all the requirements of Theorem I.17.19, with L replaced by γ'' and G by $I(c'')$.[19] It follows that the function defined by the integral (3.48), i.e., the function $G(z, w)$, is analytic on $I(c'')$ for every $w \in I(\gamma'')$, and hence on $I(c')$ for every $w \in I(\gamma')$, as required, since r'' and ρ'' can be made arbitrarily close to r' and ρ'.

Finally, to show that $G(z, w)$ is continuous on $I(c') \times I(\gamma')$, thereby completing the proof, we argue as follows: The function in the right-hand side of (3.49), regarded as a function of *three* complex variables z, w, η, is continuous on the compact six-dimensional set E defined by the relations

$$|z - z_0| \leqslant r'', \qquad |w - w_0| \leqslant \tilde{\rho} < \rho'', \qquad |\eta - w_0| = \rho'',$$

and hence is uniformly continuous on E (cf. Theorem I.3.17). Therefore, given any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\left| \frac{F(z, \eta)}{P(z, \eta)(\eta - w)} - \frac{F(z', \eta)}{P(z', w)(\eta - w')} \right| < \frac{\varepsilon}{\rho''},$$

provided that $|\eta - w_0| = \rho''$ and

$$|z - z_0| < r'', \qquad |z' - z_0| < r'', \qquad |z - z'| < \delta(\varepsilon),$$
$$|w - w_0| < \tilde{\rho}, \qquad |w' - w_0| < \tilde{\rho}, \qquad |w - w'| < \delta(\varepsilon).$$

It follows that under the same conditions,

$$|G(z, w) - G(z', w')| = \left| \frac{1}{2\pi i} \int_{\gamma''} \left[\frac{F(z, \eta)}{P(z, \eta)(\eta - w)} - \frac{F(z', \eta)}{P(z', \eta)(\eta - w')} \right] d\eta \right| < \varepsilon.$$

[19] The validity of Theorem I.17.19 is obviously unaffected by the presence of the extra parameter w (the role of w in the statement of the theorem is now played by η). The uniform boundedness of the family of functions $\Phi_w(z, \eta)$, $\eta \in \gamma''$ inside $I(c'')$ follows from the continuity of $\Phi_w(z, \eta)$ on the closed set $\overline{I(c'')} \times \gamma''$.

Thus, since r'' and ρ'' can be made arbitrarily close to r' and ρ', and since $\tilde{\rho}$ can be made arbitrarily close to ρ'', the function $G(z, w)$ is continuous on $I(c') \times I(\gamma')$, and the proof is complete.

Our next result is an immediate consequence of Weierstrass' preparation theorem:

THEOREM 3.11 (*Implicit function theorem*). *Let $F(z, w)$ be a function of two complex variables which is analytic in a neighborhood $|z - z_0| < r$, $|w - w_0| < \rho$ of the point (z_0, w_0), and suppose that*

$$F(z_0, w_0) = 0, \qquad \frac{\partial F(z_0, w_0)}{\partial w_0} \neq 0. \tag{3.50}$$

Then there are neighborhoods $\mathcal{N}(z_0)$ and $\mathcal{N}(w_0)$ such that the equation $F(z, w) = 0$ has a unique root $w = w(z)$ in $\mathcal{N}(w_0)$ for any given $z \in \mathcal{N}(z_0)$. Moreover, the function $w = w(z)$ is single-valued and analytic on $\mathcal{N}(z_0)$, and satisfies the condition $w(z_0) = w_0$.

Proof. Let $\mathcal{N}(z_0)$ and $\mathcal{N}(w_0)$ be the neighborhoods $|z - z_0| < r' < r$ and $|w - w_0| < \rho' < \rho$ constructed in the proof of Theorem 3.10. In the present case ($k = 1$), equation (3.45) reduces to

$$F(z, w) = [A_0(z) - w]G(z, w),$$

where $G(z, w) \neq 0$ on $\mathcal{N}(z_0) \times \mathcal{N}(w_0)$. Therefore $F(z, w) = 0$ is equivalent to

$$A_0(z) + w = 0, \tag{3.51}$$

or

$$w = w(z) = -A_0(z),$$

where $A_0(z)$ is single-valued and analytic on $\mathcal{N}(z_0)$. Moreover, it follows from $F(z_0, w_0) = 0$, $G(z_0, w_0) \neq 0$ that

$$w_0 = w(z_0) = -A_0(z_0).$$

Remark. In the general case ($k > 1$), we have

$$F(z_0, w_0) = \frac{\partial F(z_0, w_0)}{\partial w} = \cdots = \frac{\partial^{k-1} F(z_0, w_0)}{\partial w^{k-1}} = 0, \qquad \frac{\partial^k F(z_0, w_0)}{\partial w^k} \neq 0$$

instead of (3.50), and the equation

$$P(z, w) = A_0(z) + A_1(z)w + \cdots + A_{k-1}(z)w^{k-1} + w^k = 0$$

instead of (3.51). As we know, for every $z \in \mathcal{N}(z_0)$, this equation has k roots $w_1(z), \ldots, w_k(z)$ in $\mathcal{N}(w_0)$, some of which may be multiple roots. However, the equation $P(z, w) = 0$ can only have a multiple root if z is a point in $\mathcal{N}(z_0)$ such that both $P(z, w)$ and $(\partial/\partial w)P(z, w)$ vanish for some $w = w_j(z)$.

It is easy to see that such points are characterized by the vanishing of the *discriminant*

$$D(z) = \prod_{1 \leqslant j < l \leqslant k} [w_j(z) - w_l(z)]^2. \tag{3.52}$$

We can write (3.52) in the form of a determinant

$$D(z) = \begin{vmatrix} k & s_1 & s_2 & \cdots & s_{k-1} \\ s_1 & s_2 & s_3 & \cdots & s_k \\ \cdot & & & & \cdot \\ s_{k-1} & s_k & s_{k+1} & \cdots & s_{2k-2} \end{vmatrix} \tag{3.53}$$

(see Problem 3.15), where, as before

$$s_m = s_m(z) = w_1^m(z) + \cdots + w_k^m(z).$$

Then, since the functions $s_m(z)$ are analytic in the neighborhood $\mathcal{N}(z_0)$, as we found in the proof of Theorem 3.10, the same is true of the function $D(z)$.

Obviously $D(z_0) = 0$, since $F(z_0, w)$, and hence $P(z_0, w)$, has a zero of order $k > 1$ at the point w_0. If $D(z) \not\equiv 0$, then there is a neighborhood $\tilde{\mathcal{N}}(z_0) \subset \mathcal{N}(z_0)$ in which $D(z)$ is nonzero except at point z_0 itself. Suppose that $z_1 \in \tilde{\mathcal{N}}(z_0)$, $z_1 \neq z_0$. Then $P(z_1, w)$ has k distinct zeros, which we denote by $w_1, \ldots, w_k$, and moreover

$$\frac{\partial P(z_1, w_j)}{\partial w} \neq 0 \qquad (j = 1, \ldots, k).$$

It follows from Theorem 3.11, applied to the function $P(z, w)$, that there are neighborhoods $\mathcal{N}(z_1) \subset \mathcal{N}(z_0)$ and $\mathcal{N}_j(w_j)$ such that $P(z, w)$, and hence $F(z, w)$, has a unique zero $w = w_j(z)$ in $\mathcal{N}_j(w_k)$, for every given $z \in \mathcal{N}(z_1)$. Furthermore, the function $w_j(z)$, $j = 1, \ldots, k$, is single-valued and analytic on $\mathcal{N}(z_1)$, and satisfies the condition $w_j(z_1) = w_j$. In other words, we can define k single-valued analytic functions in the neighborhood $\mathcal{N}(z_1)$, which can be regarded as k branches of a k-valued function $w = w(z)$, defined implicitly by the equation $F(z, w)$ and taking the values $w_1, \ldots, w_k$ at the point $z = z_1$. This argument breaks down at points where the discriminant $D(z)$ vanishes, and is completely inapplicable if $D(z) \equiv 0$.

Example. Let $k = 2$, so that

$$F(z_0, w_0) = \frac{\partial F(z_0, w_0)}{\partial w} = 0, \qquad \frac{\partial^2 F(z_0, w_0)}{\partial w^2} \neq 0.$$

Suppose, for simplicity, that $z_0 = w_0 = 0$.[20] Then, according to Weierstrass'

[20] We can always achieve this by making a preliminary change of variables $z^* = z - z_0$, $w^* = w - w_0$.

preparation theorem, the equation $F(z, w) = 0$ is "locally equivalent" to the equation

$$A_0(z) + A_1(z)w + w^2 = 0, \tag{3.54}$$

where $A_0(z)$ and $A_1(z)$ are analytic in some neighborhood $\mathcal{N}(0): |z| < r'$. Moreover,

$$A_0(0) = A_1(0) = 0,$$

since (3.54) must have a double root at $z = 0$. The discriminant of (3.54) equals

$$D(z) = [A_1(z)]^2 - 4A_0(z).$$

If $D(z) \equiv 0$, we cannot apply the implicit function theorem, but then

$$A_0(z) = \tfrac{1}{4}[A_1(z)]^2,$$

and (3.54) becomes

$$[w + \tfrac{1}{2}A_1(z)]^2 = 0,$$

which implies

$$w = -\tfrac{1}{2}A_1(z).$$

Therefore, in this case, the implicit function is single-valued and analytic.

Next suppose that $D(z) \not\equiv 0$. Since $D(0) = [A_1(0)]^2 - 4A_0(0) = 0$, we can find a neighborhood $|z| < \tilde{r}$ in which $D(z)$ has no zeros except at $z = 0$ itself. Let $z_1 \neq z_0$ be a point of this neighborhood, which, to be explicit, satisfies the condition $|z_1| < \frac{1}{2}\tilde{r}$. Then $A_0(z)$, $A_1(z)$ are analytic and $D(z)$ is nonzero in the neighborhood $|z - z_1| < |z_1|$. Therefore, solving the quadratic equation (3.54) for w, we obtain two single-valued analytic branches

$$w_{1,2}(z) = -\tfrac{1}{2}A_1(z) + \tfrac{1}{2}\sqrt{D(z)} \qquad (|z - z_1| < |z_1|)$$

of the function $w = w(z)$, where each branch $w_1(z)$ and $w_2(z)$ corresponds to a different branch of the square root $\sqrt{D(z)}$. If we enlarge the neighborhood of z_1 until it includes the origin, there are two possible ways in which $w(z)$ can behave, depending on whether the zero of $D(z)$ at the origin is of even or odd order. In the first case, we have

$$D(z) = a_{2m}z^{2m} + a_{2m+1}z^{2m+1} + \cdots \qquad (a_{2m} \neq 0)$$

or

$$\sqrt{D(z)} = \sqrt{a_{2m}}z^m(1 + \cdots),$$

so that

$$w(z) = -\tfrac{1}{2}A_1(z) + \sqrt{a_{2m}}z^m(1 + \cdots).$$

This corresponds to two single-valued analytic branches of the function $w(z)$ in a neighborhood of the origin. In the second case, we have

$$D(z) = a_{2m-1}z^{2m-1} + a_{2m}z^{2m} + \cdots \qquad (a_{2m-1} \neq 0)$$

or

$$\sqrt{D(z)} = \sqrt{a_{2m-1}}z^{m-\frac{1}{2}}(1 + \cdots),$$

so that

$$w(z) = -\tfrac{1}{2}A_1(z) + \sqrt{a_{2m-1}}z^{m-\frac{1}{2}}(1 + \cdots).$$

This corresponds to one function which is double-valued in an arbitrarily small neighborhood of the origin, and in fact has an algebraic branch point of the first order at the origin.

PROBLEMS

3.1. Let $w = f(z)$ be analytic on the unit disk $K: |z| < 1$, and suppose that

$$f(0) = 0, \quad f'(0) = 1, \quad |f'(z)| \leqslant M \qquad (z \in K).$$

Prove that every point in the neighborhood

$$|w| < \frac{1}{2(M+1)} \tag{3.55}$$

has a unique inverse image in the neighborhood

$$|z| < \frac{1}{M+1}.$$

Hint. Apply Schwarz's lemma (Theorem I.17.8) to the function $f'(z) - 1$, which is bounded by $M + 1$ on K. Then use Rouché's theorem, observing that

$$f(z) - z = \int_0^z [f'(\zeta) - 1]\, d\zeta.$$

Comment. Thus, according to Theorem 3.2, $z = f^{-1}(w)$ is analytic in the neighborhood (3.55), at least.

3.2. Let $f(z) = u(x, y) + iv(x, y) \not\equiv \text{const}$ be analytic on a domain G, and let $\omega(u, v)$ be a function of two real variables which has no (relative) maxima in the uv-plane. Prove that $\omega[u(x, y), v(x, y)]$ cannot have a maximum at any point of G.

Comment. The maximum modulus principle corresponds to the case $\omega(u, v) = \sqrt{u^2 + v^2}$.

3.3. The preceding problem generalizes the maximum modulus principle. State and prove the corresponding generalization of the minimum modulus principle.

3.4. Let $f(z) \not\equiv \text{const}$ be analytic on a domain G. Prove that neither of the functions Re $f(z)$, Im $f(z)$ can have a maximum or a minimum at a point of G.

3.5. Expand e^z in powers of w, given that $(z + 1)^2 w = z$.

3.6. Expand e^{-z} in powers of w, given that $zw = z - a$.

3.7. Use Lagrange's series to find the two branches of the function $z = f^{-1}(w)$, if $w = f(z) = 2z + z^2$, where $f^{-1}(0) = 0$ for one branch and $f^{-1}(0) = -2$ for the other.

Hint. Although $z = f^{-1}(w)$ is double-valued, there is no need to use Theorem 3.6, since $f(z)$ has a *simple* zero at each of the points $z = 0$ and $z = -2$.

3.8. Find $z = f^{-1}(w)$ if $w = f(z) = 2z^2 + z^3$.

Hint. Theorem 3.6 is now required.

3.9. Prove that the Legendre polynomials satisfy the relations

$$P_n(t) = P'_{n+1}(t) - 2tP'_n(t) + P'_{n-1}(t),$$
$$(2n + 1)P_n(t) = P'_{n+1}(t) + P'_{n-1}(t)$$

for all $n \geqslant 1$.

Hint. Differentiate (3.25) term by term with respect to t.

3.10. The polynomials $H_n(t)$ defined by the relation

$$e^{2tw - w^2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} w^n$$

[cf. (3.25)] are called the *Hermite polynomials.* Prove that

a) $H'_n(t) = 2nH_{n-1}(t) \qquad (n \geqslant 1);$
b) $H_{n+1}(t) - 2tH_n(t) + 2nH_{n-1}(t) \qquad (n \geqslant 1);$
c) $H''_n(t) - 2tH'_n(t) + 2nH_n(t) = 0 \qquad (n \geqslant 0);$
d) $H_n(t) = (-1)^n e^{t^2} \dfrac{d^n(e^{-t^2})}{dt^n}.$

Hint. If $F(t, w) = e^{2tw - w^2}$, then

$$\frac{\partial F}{\partial t} = 2wF, \qquad \frac{\partial F}{\partial w} = (2t - 2w)F.$$

3.11. Verify that $\delta(\rho_0) = \sqrt{\rho_0^2 - 1}$ is actually the radius of convergence of the series (3.31), where $\coth \rho_0 = \rho_0$.

Hint. As shown on p. 100, the radius of convergence of (3.31) cannot be smaller than $\delta(\rho_0)$. Moreover, no point $w = f(z)$ such that $f'(z) = 0$, i.e.,

$$\tan z = z - \frac{\pi}{2}, \tag{3.56}$$

can lie inside the circle of convergence. But (3.56) has the root

$$z = -i\rho_0 + \frac{\pi}{2},$$

leading to the value

$$\left| f\left(-i\rho_0 + \frac{\pi}{2}\right) \right| = \left| \frac{-i\rho_0}{\cosh \rho_0} \right| = \delta(\rho_0).$$

Therefore the radius of convergence of (3.31) cannot be larger than $\delta(\rho_0)$.

3.12. Show that the series (3.39) is absolutely convergent on the domain $I(c_r) \times I(\gamma_\rho)$ and uniformly convergent on every compact subset of $I(c_r) \times I(\gamma_\rho)$.

3.13. By properly grouping terms, prove that the Taylor series (3.43) can be written in either of the two forms

$$F(z, w) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k F(z_0, w)}{\partial z^k} (z - z_0)^k,$$

$$F(z, w) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m F(z, w_0)}{\partial w^m} (w - w_0)^m.$$

Comment. The same results can be obtained directly by regarding $F(z, w)$ as an analytic function of z for fixed w, or as an analytic function of w for fixed z.

3.14. Prove formula (3.46).

Hint. Given k variables $w_1, w_2, \ldots, w_k$ and $j \leqslant k$ fixed positive integers $n_1, n_2, \ldots, n_j$, let $S(w_1^{n_1} w_2^{n_2} \cdots w_j^{n_j})$ denote the sum of all terms obtained from $w_1^{n_1} w_2^{n_2} \cdots w_j^{n_j}$ by replacing the variables $w_1, w_2, \ldots, w_j$ by any other set of variables $w_{l_1}, w_{l_2}, \ldots, w_{l_j}$ whose indices are arranged in increasing order, i.e.,

$$S(w_1^{n_1} w_2^{n_2} \cdots w_j^{n_j}) = \sum_{1 \leqslant l_1 < l_2 < \cdots < l_j \leqslant k} w_{l_1}^{n_1} w_{l_2}^{n_2} \cdots w_{l_j}^{n_j}.$$

Thus, for example,

$$S(w_1^m) = w_1^m + \cdots + w_k^m,$$

which is just the quantity we denote by s_m (the dependence on z is suppressed for simplicity),

$$S(w_1 w_2) = w_1 w_2 + w_1 w_3 + \cdots + w_{k-1} w_k,$$

and so on. Then it is easy to see that

$$\begin{aligned}
s_{m-1} A_{k-1} &= -s_m - S(w_1^{m-1} w_2), \\
s_{m-2} A_{k-2} &= S(w_1^{m-1} w_2) + S(w_1^{m-2} w_2 w_3), \\
&\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
s_1 A_{k-m+1} &= (-1)^{m-1} S(w_1^2 w_2 \cdots w_{m-1}) - m A_{k-m},
\end{aligned}$$

and the desired result is obtained by adding all these equations.

3.15. Prove formula (3.53).

Hint. Calculate the square of the Vandermonde determinant[21]

$$\prod_{1 \leqslant j < l \leqslant k} (w_j - w_l) = \begin{vmatrix} 1 & 1 & & 1 \\ w_1 & w_2 & & w_k \\ w_1^2 & w_2^2 & & w_k^2 \\ \cdot & \cdot & & \cdot \\ w_1^{k-1} & w_2^{k-1} & \cdots & w_k^{k-1} \end{vmatrix}.$$

3.16. Investigate the function $w = w(z)$ defined as the solution of the equation $\sin zw = 0$.

3.17. Investigate the function $z = z(w)$ defined as the solution of the equation $e^{zw} = 1$.

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[21] See e.g., G. E. Shilov, *An Introduction to the Theory of Linear Spaces* (translated by R. A. Silverman), Prentice-Hall, Inc., Englewood Cliffs, N. J. (1963), p. 15.

4

UNIVALENT FUNCTIONS

17. Some Elementary Results

It will be recalled from Sec. I.53 that a single-valued function $f(z)$ is said to be *univalent* (synonymously, *schlicht* or *simple*) on a domain G if $f(z)$ is wide-sense continuous on G and analytic on G except possibly at poles,[1] and if $f(z)$ takes distinct values at distinct points of G, i.e.,

$$f(z_1) \neq f(z_2) \quad \text{if} \quad z_1 \neq z_2 \quad (z_1, z_2 \in G).$$

(It follows at once that a univalent function can have only one pole.) On the other hand, if there are at least two distinct points $z_1, z_2 \in G$ such that $f(z_1) = f(z_2)$, then $f(z)$ is said to be *multivalent* on G. Moreover, $f(z)$ is said to be *k-valent* on G if every value of $f(z)$ assumed at a point of G is assumed at precisely $k > 1$ distinct points of G. Using this language, we can paraphrase Theorem 3.4, Corollary 3 as follows, elevating it to the status of a theorem:

THEOREM 4.1. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a w_0-point of order $k \geqslant 1$. Then there is a neighborhood of z_0 in which $f(z)$ is univalent if $k = 1$, and a deleted neighborhood of z_0 in which $f(z)$ is k-valent if $k > 1$.*

COROLLARY 1. *If $w = f(z)$ is univalent on a domain G, then all the points of G are simple w-points. In particular, the zero and pole of $f(z)$ in G (if such exist) are simple.*

[1] Things were put somewhat differently in Sec. I.53, where the concept of a pole was not yet available (except in the special case of rational functions). We now see that the exceptional set I on p. I.213 can only consist of poles of $f(z)$, since $f(z)$ is assumed to be wide-sense continuous on G. In fact, I can only consist of a single pole if $f(z)$ is univalent.

COROLLARY 2. *If $w = f(z)$ is univalent on a domain G, then $f'(z)$ is nonzero at every point of G where $f'(z)$ exists.*[2]

COROLLARY 3. *If $w = f(z)$ is univalent on a domain G, then the mapping $w = f(z)$ is conformal at every point of G.*

Proof. Since $f'(z) \neq 0$ for every $z \in G$ (except at a possible pole z_0), we need only use Theorem I.8.2, referring to Remarks 1 and 2 of Sec. I.34 for the cases where z_0 or w_0 (or both) are infinite.

Remark. Let $f(z) = e^z$, and let G be the finite z-plane. Then $f'(z) = e^z$ is nonzero at every point of G, so that $f(z)$ is conformal at every point of G. However, $f(z)$ is not univalent on G, since $f(z + 2\pi i) = f(z)$ for every $z \in G$. Therefore the converses of Corollaries 2 and 3 are false.

In passing we note that conformality must be destroyed at a point where $f'(z)$ vanishes:

THEOREM 4.2. *Let the function $f(z)$ be analytic at the point $z = z_0$, where it has a w_0-point of order $k > 1$. Then, under the mapping $w = f(z)$, every angle with its vertex at z_0 is enlarged k times.*

Proof. The proof is identical with that of the first half of Theorem I.9.2 [for the special case where $f(z)$ is a polynomial], except that now instead of (I.9.4), we have

$$f(z) - f(z_0) = (z - z_0)^k(a_k + a_{k+1} + \cdots),$$

where the expression in brackets does not vanish for $z = z_0$, since

$$a_k = \frac{f^{(k)}(z_0)}{k!} \neq 0.$$

The cases $z_0 = \infty$, $w_0 = \infty$ and $z_0 = w_0 = \infty$ are handled in the usual way.

Returning to univalent functions, we have the following important

THEOREM 4.3. *If $f(z)$ is univalent in the whole finite z-plane, then $f(z)$ is a Möbius transformation, i.e., $f(z)$ is of the form*

$$f(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0). \tag{4.1}$$

Proof.[3] *A priori*, the point at infinity can be a regular point, a pole or an essential singular point. However, the third possibility is immediately excluded. In fact, let $\mathcal{K} = f(K)$, where K is the unit

[2] This last condition allows for the possibility of $f(z)$ having a pole z_0 in G. Recall from Sec. I.34 that $f'(\infty)$ is defined as $f^*(0)$, where $f^*(\zeta) = f(1/\zeta)$.

[3] Note that much of the proof is unnecessary if we assume from the outset that $f(z)$ is univalent in the *extended* z-plane.

disk $|z| < 1$. According to Theorem 3.5, $\mathscr{K}$ is a domain. Let w_0 be an arbitrary point of $\mathscr{K}$, with a neighborhood $\mathscr{N}(w_0) \subset \mathscr{K}$. Being univalent, $f(z)$ cannot take any value in $\mathscr{N}(w_0)$ at a point belonging to the complement of K. Therefore, there cannot exist a sequence $\{z_n\}$ converging to ∞ such that $f(z_n) \to w_0$ as $z_n \to \infty$. It follows from Theorem 1.8 that $f(z)$ cannot have an essential singular point at ∞. Therefore ∞ is either a regular point or a pole of $f(z)$.

If ∞ is a pole, it must be a simple pole, since otherwise, by Theorem 3.4, Corollary 5, there would be a neighborhood $\mathscr{N}(\infty)$ in which every value of $f(z)$ except ∞ assumed at a point of $\mathscr{N}(\infty)$ is assumed at k distinct points of $\mathscr{N}(\infty)$, where $k > 1$ is the order of the pole. But this is impossible, since $f(z)$ is univalent. Moreover, if ∞ is a pole, no finite point z_1 can be a pole of $f(z)$. In fact, suppose $f(z)$ has a pole at $z_1 \neq \infty$, and let $\mathscr{K}_1 = f(K_1)$, where K_1 is the disk $|z - z_1| < 1$. Then the domain $\mathscr{K}_1$ contains the point ∞ and some neighborhood $\mathscr{N}(\infty)$. Being univalent, $f(z)$ cannot take any value in $\mathscr{N}(\infty)$ at a point belonging to the complement of K_1, i.e., $f(z)$ is bounded at ∞, contrary to the assumption that ∞ is a pole.

Therefore, $f(z)$ can have no singular points in the extended z-plane, except for one simple pole. Moreover, $f(z)$ must have such a pole, since otherwise $f(z) \equiv$ const by Liouville's theorem (Theorem I.16.8), contrary to the assumption that $f(z)$ is univalent. If the pole is at infinity, then $f(z)$ is an entire function with a simple pole at infinity, i.e., $f(z)$ is the entire linear transformation

$$f(z) = az + b \qquad (a \neq 0)$$

[cf. p. I.122], corresponding to $c = 0$, $d = 1$ in (4.1). If the pole is at a finite point z_1, and if

$$\frac{a_1}{z - z_1}$$

is the principal part of the corresponding Laurent expansion, then

$$f(z) - \frac{a_1}{z - z_1} \qquad (a_1 \neq 0)$$

has no singular points in the extended z-plane. It follows from Liouville's theorem that

$$f(z) - \frac{a_1}{z - z_1} \equiv \text{const},$$

which implies that $f(z)$ is a Möbius transformation, with $c = 1$, $ad - bc = -a_1 \neq 0$ in (4.1). This completes the proof.

The most elementary examples show that the sum, difference, product and quotient of two functions univalent on a given domain G need not be

univalent on G, and the same is true of the derivative and integral of a function univalent on G. However, if $\varphi(z)$ and $f(z)$ are univalent, then obviously so is the composite function $\varphi[f(z)]$, provided that $\varphi(z)$ is defined on the range of $f(z)$. Briefly, we say that *the composition of two univalent functions is univalent.*

THEOREM 4.4. *If the sequence $\{f_n(z)\}$ is uniformly convergent on every compact subset of a domain G, and if every term $f_n(z)$ is univalent on G, then the limit function*

$$f(z) = \lim_{n \to \infty} f_n(z)$$

is also univalent on G, unless $f(z) \equiv$ const.

Proof. The fact that $f(z)$ is analytic on G is an immediate consequence of Weierstrass' theorem on uniformly convergent sequences of analytic functions (Theorem I.15.8).[4] Therefore we need only prove that

$$f(z_1) \neq f(z_2) \quad \text{if} \quad z_1 \neq z_2 \qquad (z_1, z_2 \in G).$$

Suppose the opposite, i.e., let $f(z_1) = f(z_2) = a$, where $z_1 \neq z_2$ and $z_1, z_2 \in G$. Then the function $f(z) - a$ which is the limit of the uniformly convergent sequence $\{f_n(z) - a\}$ has at least two zeros in G. Therefore, according to Hurwitz's theorem (Theorem 2.4), every function $f_n(z) - a$, starting from some sufficiently large value of n, must have at least one zero each in an arbitrarily small neighborhood of z_1 and in an arbitrarily small neighborhood of z_2. Choosing these neighborhoods so small that they are disjoint, we conclude that every function $f_n(z)$, starting from some value of n, takes the same value at two distinct points of G (at least). This contradicts the assumption that every function $f_n(z)$ is univalent on G, thereby proving the theorem.

18. Sufficient Conditions for Univalence

In this section we shall prove some simple theorems establishing criteria for a function to be univalent. Our first result along these lines is

THEOREM 4.5. *Given a closed rectifiable Jordan curve γ, suppose $f(z)$ is analytic on $\overline{I(\gamma)}$ and one-to-one on γ. Then $f(z)$ is univalent on $I(\gamma)$, which it maps onto $I(\Gamma)$, where $\Gamma = f(\gamma)$.*[5]

Proof. First we note that Γ is itself a closed rectifiable Jordan curve (cf. Problem I.6.6 and Theorem I.14.9), while $f[I(\gamma)]$ is a domain (see

[4] Note that there is an integer $N > 0$ such that $f_n(z)$ has no pole in G if $n > N$.

[5] Clearly, the word "univalent" can be replaced by "one-to-one" in the statement of Theorem 4.5 (and the other theorems of this section).

Theorem 3.5). Let w_1 be an arbitrary point of $I(\Gamma)$. According to the argument principle (Theorem 2.3), the number of w_1-points of $f(z)$ inside γ minus the number of poles of $f(z)$ inside γ equals ν, the number of circuits around the point w_1 made by the point $w = f(z)$ as the point z traverses γ once in the positive direction. *A priori*, ν can only equal -1 or $+1$, since Γ is a closed Jordan curve (cf. footnote 5, p. I.219). But $f(z)$ has no poles inside γ, and hence ν must equal $+1$, i.e., $f(z)$ takes every value $w_1 \in I(\Gamma)$ once, and we have incidentally proved that $w = f(z)$ traverses Γ in the same direction as z traverses γ. It follows that

$$f[I(\gamma)] \supset I(\Gamma). \tag{4.2}$$

Next let w_2 be an arbitrary point of $E(\Gamma)$. Then, as z traverses the curve γ, the point $w = f(z)$ traverses the curve Γ and obviously cannot wind around w_2. Again applying the argument principle, and the fact that $f(z)$ has no poles inside γ, we find that $f(z)$ has no w_2-points inside γ, i.e., $E(\Gamma) \cap f[I(\gamma)] = 0$. We also have $\Gamma \cap f[I(\gamma)] = 0$, since $w \in \Gamma$, $w \in f[I(\gamma)]$ implies that some neighborhood $\mathcal{N}(w)$ is contained in $f[I(\gamma)]$. But this contradicts what was just proved, since every neighborhood of a point of Γ must contain points of $E(\Gamma)$. Therefore

$$f[I(\gamma)] \subset I(\Gamma), \tag{4.3}$$

and, together, (4.2) and (4.3) imply the desired result $f[I(\gamma)] = I(\Gamma)$. Moreover, as already noted, every point of $I(\Gamma)$ has precisely one inverse image in $I(\gamma)$. It follows that the mapping from $I(\gamma)$ to $I(\Gamma)$ is one-to-one, i.e., $f(z)$ is univalent on $I(\gamma)$, as asserted.

There is an analogous theorem for the exterior of γ:

THEOREM 4.6. *Given a closed rectifiable Jordan curve γ, suppose $f(z)$ is analytic on $\overline{E(\gamma)}$ and one-to-one on γ. Then $f(z)$ is univalent on $E(\gamma)$, which it maps onto $I(\Gamma)$, where $\Gamma = f(\gamma)$.*

Proof. Let z_0 be any point in $I(\gamma)$. Then the transformation

$$\zeta = \frac{1}{z - z_0} \tag{4.4}$$

carries γ into a closed rectifiable Jordan curve γ^* in the ζ-plane, and maps $E(\gamma)$ onto $I(\gamma^*)$. We now write

$$f^*(\zeta) = f\left(z_0 + \frac{1}{\zeta}\right), \tag{4.5}$$

noting that $f^*(\zeta)$ is analytic on $\overline{I(\gamma^*)}$ and one-to-one on γ^*, which it maps onto Γ. Applying Theorem 4.5 to the curve γ^* and the function

$f^*(\zeta)$, we see that $f^*(\zeta)$ is univalent on $I(\gamma^*)$, which it maps onto $I(\Gamma)$. Therefore $f(z)$ is univalent on $E(\gamma)$, which it maps onto $I(\Gamma)$.[6]

Next we allow the curve γ to be *unbounded*, i.e., to pass through the point at infinity [cf. p. I.333]:

THEOREM 4.7. *Given a closed unbounded Jordan curve γ, let D_1 and D_2 be the two domains with boundary γ.*[7] *Suppose $f(z)$ is analytic on $\bar{D}_1$ and one-to-one on γ, and let γ^* be the closed Jordan curve which is the image of γ under the transformation* (4.4), *where z_0 is an arbitrary point of D_2. Then $f(z)$ is univalent on D_1, which it maps onto $I(\Gamma)$, where $\Gamma = f(\gamma)$, provided that γ^* is rectifiable.*

Proof. Whether or not γ^* is rectifiable does not depend on the choice of the point $z_0 \in D_2$, since if $\tilde{\gamma}$ is the image of γ under the transformation

$$\zeta = \frac{1}{z - z_1} \qquad (z_1 \in D_2,\ z_1 \neq z_0)$$

and if $\tilde{\gamma}$ is rectifiable, then so is the curve obtained by subjecting $\tilde{\gamma}$ to the transformation

$$Z = \frac{z}{(z_1 - z_0)z + 1}. \tag{4.6}$$

But it is easy to see that (4.6) carries $\tilde{\gamma}$ into γ^*. Noting that (4.4) maps D_1 onto $I(\gamma^*)$, we apply Theorem 4.5 to the curve γ^* and the function (4.5). It follows at once that $f^*(\zeta)$ is univalent on $I(\gamma^*)$, which it maps onto $I(\Gamma)$, and hence $f(z)$ is univalent on D_1, which it maps onto $I(\Gamma)$.

We now generalize Theorem 4.5 further, by relaxing somewhat the requirement that $f(z)$ be analytic on $\overline{I(\gamma)}$:

THEOREM 4.8. *Given a closed rectifiable Jordan curve γ, suppose $f(z)$ is analytic on $\overline{I(\gamma)}$ except at a finite number of points of γ, continuous on $\overline{I(\gamma)}$ and one-to-one on γ. Then $f(z)$ is univalent on $I(\gamma)$, which it maps onto $I(\Gamma)$, where $\Gamma = f(\gamma)$.*

Proof. Let $z_1, \ldots, z_n$ denote the exceptional points of γ, i.e., the points at which $f(z)$ fails to be analytic. With the points $z_1, \ldots, z_n$ as centers, we draw circles $|z - z_k| = \rho$, choosing the radius ρ so small that all n circles are disjoint. Each circle has at least one arc contained in $I(\gamma)$, except for its end points which lie on γ itself. Let $\sigma_{k,\rho}$ denote

[6] In arguments like this, we repeatedly use the fact that the composition of two univalent functions is univalent (see p. 118), and the obvious univalence of functions like (4.4) and its inverse.

[7] Think in terms of the Riemann sphere, with $\infty \in \gamma$ (see Chap. I.5). Note that it no longer makes sense to talk about the interior (or exterior) of γ.

an arc of the circle $|z - z_k| = \rho$ which, together with a suitable arc $\gamma_{k,\rho}$ of the curve γ (with the same end points as $\sigma_{k,\rho}$) forms the boundary of a domain $D_{k,\rho}$ such that $D_{k,\rho} \subset I(\gamma)$, while z_k belongs to the boundary of $D_{k,\rho}$ (see Figure 4.1). We also subject the arcs $\sigma_{k,\rho}$ and $\gamma_{k,\rho}$ to the additional requirement that none of the domains $D_{k,\rho}$ contain a fixed point $\tilde{z} \in I(\gamma)$.

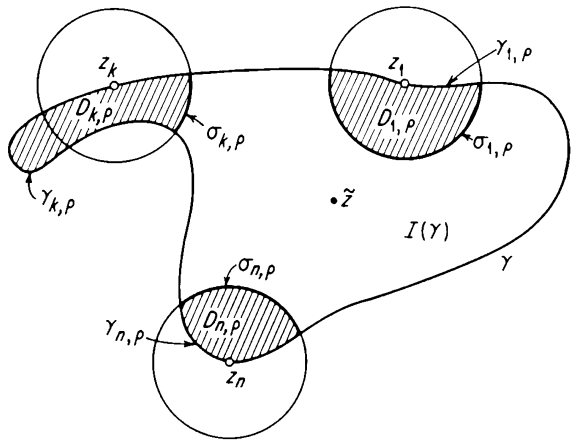

FIGURE 4.1

It is not hard to see that as $\rho \to 0$, the arcs $\sigma_{k,\rho}$ and $\gamma_{k,\rho}$, as well as the domains $D_{k,\rho}$ bounded by them, eventually lie in an arbitrarily small neighborhood of the point z_k. Thus we now choose ρ so small that not only the disks $|z - z_k| \leqslant \rho$ but also the closed domains $\overline{D}_{k,\rho}$ are disjoint. Then we delete the arcs $\gamma_{k,\rho}$ from the curve γ, and replace them by the circular arcs $\sigma_{k,\rho}$. This has the effect of replacing γ by a new closed rectifiable Jordan curve γ_ρ, such that $f(z)$ is analytic on $\overline{I(\gamma_\rho)}$, with no exceptional points. Moreover $I(\gamma_\rho) \subset I(\gamma)$, and $\Gamma_\rho = f(\gamma_\rho)$ is obtained from $\Gamma = f(\gamma)$ by deleting the arcs $\Gamma_{k,\rho} = f(\gamma_{k,\rho})$ and replacing them by the arcs $\Sigma_{k,\rho} = f(\sigma_{k,\rho})$. It should be noted that we cannot assert that the correspondence between γ_ρ and its image Γ_ρ is one-to-one, since it is only known that the correspondence between γ and Γ is one-to-one. Therefore there is no possibility of applying Theorem 4.5, and in fact, we shall have to retrace the steps leading to the proof of Theorem 4.5.

Now let

$$w_k = \lim_{\substack{z \to z_k \\ z \in \overline{I(\gamma)}}} f(z) \qquad (k = 1, \ldots, n),$$

and let w' be an arbitrary point of $I(\Gamma)$. If we choose disjoint neighborhoods $\mathcal{N}(w_k)$ so small that $w' \notin \mathcal{N}(w_k)$, $k = 1, \ldots, n$, then, for sufficiently small ρ, the arcs $\Gamma_{k,\rho}$ and $\Sigma_{k,\rho}$ will be contained in $\mathcal{N}(w_k)$.

Obviously, the change in Arg $[f(z) - w']$ as z describes the curve γ_ρ once in the positive direction equals the sum of the changes in Arg $[f(z) - w']$ caused by traversing the arcs of Γ which remain after deleting the arcs $\Gamma_{k,\rho}$ plus the sum of the changes in Arg $[f(z) - w']$ caused by traversing the arcs $\Sigma_{k,\rho}$. But $\Gamma_{k,\rho}$ and $\Sigma_{k,\rho}$ have common end points, and are contained in the neighborhood $\mathcal{N}(w_k)$. Therefore, since $w' \notin \mathcal{N}(w_k)$, we must have[8]

$$\operatorname*{Var}_{z \in \sigma_{k,\rho}} \operatorname{Arg} [f(z) - w'] = \operatorname*{Var}_{z \in \gamma_{k,\rho}} \operatorname{Arg} [f(z) - w']$$

(concerning the notation, see footnote 3, p. 48). Therefore

$$\operatorname*{Var}_{z \in \gamma_\rho} \operatorname{Arg} [f(z) - w'] = \operatorname*{Var}_{z \in \gamma} \operatorname{Arg} [f(z) - w'] = 2\pi,$$

as in the proof of Theorem 4.5, i.e., for all sufficiently small ρ, $f(z)$ takes the value w' inside γ_ρ, in fact, just once. The same must be true for the curve γ, since every $I(\gamma_\rho) \subset I(\gamma)$.

Finally, let w'' be an arbitrary point of $E(\Gamma)$. Then, by the same reasoning,

$$\operatorname*{Var}_{z \in \gamma_\rho} \operatorname{Arg} [f(z) - w''] = \operatorname*{Var}_{z \in \gamma} \operatorname{Arg} [f(z) - w''] = 0,$$

i.e., for all sufficiently small ρ, $f(z)$ does not take the value w'' inside γ_ρ. Moreover, just as in the proof of Theorem 4.5, the same is true if $w'' \in \Gamma$. Thus $f(z)$ takes no value in $E(\Gamma) \cup \Gamma$ at a point of $I(\gamma_\rho)$, and since every point $z \in I(\gamma)$ belongs to all the domains $I(\gamma_\rho)$, starting from some sufficiently small value of $\rho = \rho(z)$, the function $f(z)$ takes no value in $E(\Gamma) \cup \Gamma$ at a point of $I(\gamma)$. The proof follows at once, if we combine this with the fact (just proved) that every point of $I(\Gamma)$ has exactly one inverse image in $I(\gamma)$.

COROLLARY 1. *Given a closed rectifiable Jordan curve γ, suppose $f(z)$ is analytic on $\overline{E(\gamma)}$ except at a finite number of points of γ, continuous on $\overline{E(\gamma)}$ and one-to-one on γ. Then $f(z)$ is univalent on $E(\gamma)$, which it maps onto $I(\Gamma)$, where $\Gamma = f(\gamma)$.*

Proof. Follow the proof of Theorem 4.6.

COROLLARY 2. *Given a closed unbounded Jordan curve γ, let D_1 and D_2 be the two domains with boundary γ. Suppose $f(z)$ is analytic on $\bar{D}_1$ except at a finite number of points of γ, continuous on $\bar{D}_1$ and one-to-*

[8] This is an immediate consequence of the fact that on the one hand

$$\operatorname*{Var}_{z \in \sigma_{k,\rho}} \operatorname{Arg} [f(z) - w'] - \operatorname*{Var}_{z \in \gamma_{k,\rho}} \operatorname{Arg} [f(z) - w'] = 2k\pi$$

for some $k = 0, \pm 1, \pm 2, \ldots$, while on the other hand,

$$\left| \operatorname*{Var}_{z \in \sigma_{k,\rho}} \operatorname{Arg} [f(z) - w'] \right| < \pi, \quad \left| \operatorname*{Var}_{z \in \gamma_{k,\rho}} \operatorname{Arg} [f(z) - w'] \right| < \pi.$$

one on γ. *Let* γ^* *be the closed Jordan curve which is the image of* γ *under the transformation* (4.4), *where* z_0 *is an arbitrary point of* D_2. *Then* $f(z)$ *is univalent on* D_1, *which it maps onto* $I(\Gamma)$, *where* $\Gamma = f(\gamma)$, *provided that* γ^* *is rectifiable.*

Proof. Follow the proof of Theorem 4.7.

So far, we have only allowed γ to pass through the point at infinity. Things become more complicated when $f(z)$ becomes infinite at a point of γ, i.e., if $\Gamma = f(\gamma)$ passes through the point at infinity, as shown by the following

Example. Consider the function

$$f(z) = i\left(\frac{z-1}{z+1}\right)^3,$$

with a pole of order 3 at the point $z = -1$, and let γ be the unit circle. Since $z = e^{i\theta}$ $(-\pi < \theta \leqslant \pi)$ on γ, we have

$$f(z) = f(e^{i\theta}) = i\left(\frac{e^{i\theta}-1}{e^{i\theta}+1}\right)^3 = \tan^3\frac{\theta}{2}.$$

Therefore $f(z)$ is one-to-one on γ, which it maps onto the real axis Γ in the w-plane. To find the image of the unit disk $I(\gamma)$ under the mapping $w = f(z)$ we make the preliminary transformation

$$\zeta = -i\frac{z-1}{z+1},$$

corresponding to a one-to-one mapping of $I(\gamma)$ onto the finite right half-plane D: $\operatorname{Re}\zeta > 0$. As a result, the function $w = f(z)$ goes into $w = f^*(\zeta) = \zeta^3$. But $f^*(\zeta)$ obviously fails to be univalent on D, and in fact maps D onto the extended w-plane minus the points 0 and ∞. Therefore $f(z)$ fails to be univalent on $I(\gamma)$, which it maps onto the extended w-plane minus the points 0 and ∞, i.e., *onto a domain whose boundary is not* $\Gamma = f(\gamma)$!

The example shows that as it stands, Theorem 4.8 does not apply to the case where $f(z)$ becomes infinite at a point of γ, unless extra conditions are imposed on $f(z)$, as in

THEOREM 4.9. *Given a closed rectifiable Jordan curve* γ, *suppose* $f(z)$ *is analytic on* $\overline{I(\gamma)}$ *except at a finite number of points of* γ, *at one of which* $f(z)$ *becomes infinite. Moreover, suppose* $f(z)$ *is wide-sense continuous on* $\overline{I(\gamma)}$ *and one-to-one on* γ, *and suppose there is a point* w_0 *which does not belong to* $f[\overline{I(\gamma)}]$. *Then* $f(z)$ *is univalent on* $I(\gamma)$, *which it maps onto one of the domains with boundary* $\Gamma = f(\gamma)$,[9] *i.e., the*

[9] Note that Γ necessarily passes through the point at infinity.

domain on the left of an observer moving with the point $w = f(z)$ as z traverses γ once in the positive direction.

Proof. The function

$$\varphi(z) = \frac{1}{f(z) - w_0}$$

is continuous on $\overline{I(\gamma)}$ in the ordinary sense, and one-to-one on γ. In particular, the curve $\Gamma^* = \varphi(\gamma)$ does not pass through infinity, and is in fact a closed rectifiable Jordan curve. Moreover, $\varphi(z)$ satisfies all the other conditions of Theorem 4.8. It follows that $\varphi(z)$ is univalent on $I(\gamma)$, which it maps onto $I(\Gamma^*)$. Therefore $f(z)$ is univalent on $I(\gamma)$, which it maps onto one of the domains with boundary Γ. The correctness of the characterization of this domain follows from the corresponding property of Möbius transformations, applied to the mapping from Γ^* to Γ (cf. p. I.177).

19. Mapping of the Upper Half-Plane onto a Rectangle

To illustrate the application of the results of the preceding section, consider the function

$$w = f(z) = \int_0^z \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}} \qquad (0 < k < 1), \tag{4.7}$$

known as the *elliptic integral of the first kind (in Legendre's form)*, where the path of integration is any rectifiable curve joining the points 0 and z. The function

$$\sqrt{(1 - t^2)(1 - k^2t^2)} \tag{4.8}$$

is double-valued, and has first-order branch points at $t = \pm 1$ and $t = \pm 1/k$. Since these points all lie on the real axis, we can define two single-valued analytic branches of (4.8) in the z-plane cut along the segments $-1/k \leqslant x \leqslant 1$ and $1 \leqslant x \leqslant 1/k$ of the real axis, in particular, in the upper half-plane D_1: Im $z > 0$. Of these two branches which take values with opposite signs at every point, we choose the one which takes positive values when t belongs to the interval $(0, 1)$.[10] In a neighborhood $|z| > 1/k$ of the point at infinity, the integrand in (4.7) has the expansion

$$\begin{aligned}(1 - t^2)^{-1/2}(1 - k^2t^2)^{-1/2} &= k^{-1}t^{-2}(1 - t^{-2})(1 - k^{-2}t^{-2})^{-1/2} \\ &= k^{-1}t^{-2}(1 + \tfrac{1}{2}t^{-2} + \cdots)\left(1 + \frac{1}{2k^2}\,t^{-2} + \cdots\right) \\ &= \pm[k^{-1}t^{-2} + \tfrac{1}{2}(k^{-1} + k^{-3})t^{-4} + \cdots],\end{aligned}$$

[10] This is in keeping with what we mean by $\sqrt{a}$ when $a > 0$, i.e., the *positive* square root.

which implies

$$f(z) = C \mp \left[k^{-1}z^{-1} + \frac{1}{2\cdot 3}(k^{-1} + k^{-3})z^{-3} + \cdots\right],$$

where C is a constant. Therefore it is clear that both branches of $f(z)$ are analytic at infinity. Moreover, both branches are continuous on $\bar{D}_1$.

With a view to eventually applying Theorem 4.8, Corollary 2, we now identify the real axis with the curve γ and the lower half-plane $\operatorname{Im} z < 0$ with D_2.[11] Consider the image of γ under the mapping (4.7). On the interval $0 < x < 1$, we have

$$w = f(x) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}},$$

which is positive and increases from 0 to

$$K = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}$$

as x goes from 0 to 1. In the interval $1 < x < 1/k$, the integrand in (4.7) becomes

$$\frac{1}{\pm i\sqrt{(t^2 - 1)(1 - k^2t^2)}}. \tag{4.9}$$

Here the sign cannot be chosen arbitrarily, but must agree with our previous choice of the branch of (4.8), in order to guarantee continuity of this branch on $\bar{D}_1$. Writing $(1 - t^2)(1 - k^2t^2)$ in the form

$$k^2(t - 1)[t - (-1)]\left(t - \frac{1}{k}\right)\left[t - \left(-\frac{1}{k}\right)\right] = g(t), \tag{4.10}$$

we observe that in going from the interval $0 < x < 1$ to the interval $1 < x < 1/k$, the change in $\operatorname{Arg} g(t)$ equals $-\pi$.[12] In fact, the change in $\operatorname{Arg} g(t)$ equals the sum of the changes in arguments of all the separate factors in (4.10), but none of these arguments changes except $\operatorname{Arg}(t - 1)$, which changes by $-\pi$. Therefore the change in $\operatorname{Arg}\sqrt{g(t)}$ is $-\pi/2$, so that $\operatorname{Arg}\sqrt{g(t)}$ becomes

$$-\frac{\pi}{2} + 2k\pi, \tag{4.11}$$

---

[11] Note that if z_0 is any point of D_2, then, as required, $\zeta = 1/(z - z_0)$ maps the real axis γ onto a rectifiable curve, in fact a circle. For example, $\zeta = 1/(z + i)$ maps γ onto the circle $|z - \frac{1}{2}| = \frac{1}{2}$.

[12] This is easily seen by drawing a small semicircle in the upper half-plane with center $t = 1$, and examining the behavior of $\operatorname{Arg} g(t)$ as t traverses the semicircle going from left to right.

since Arg $\sqrt{g(t)} = 2k\pi$ for $0 < t < 1$. Thus we see that of the two values $\pm i\sqrt{(t^2 - 1)(1 - k^2t^2)}$ in (4.9), the correct choice is $-i\sqrt{(t^2 - 1)(1 - k^2t^2)}$. It follows that

$$\begin{aligned} w = f(x) &= \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}} \\ &= \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}} + i\int_1^x \frac{dt}{\sqrt{(t^2 - 1)(1 - k^2t^2)}} \\ &= K + i\int_1^x \frac{dt}{\sqrt{(t^2 - 1)(1 - k^2t^2)}} \end{aligned}$$

for $1 < x < 1/k$. Therefore, as the point $z = x$ traverses the interval $1 < x < 1/k$, the image point $w = f(x)$ traverses the line segment parallel to the imaginary axis going from the point K to the point $K + iK'$, where

$$K' = \int_1^{1/k} \frac{dt}{\sqrt{(t^2 - 1)(1 - k^2t^2)}}.$$

By making the substitution

$$t = \frac{1}{\sqrt{1 - k'^2\tau^2}} \qquad (k'^2 = 1 - k^2),$$

and then changing τ back to t, we can write the integral for K' in the same form as the integral for K:

$$K' = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k'^2t^2)}} \qquad (k'^2 = 1 - k^2).$$

Next we consider the interval $1/k < x < +\infty$. In this interval, the integrand in (4.7) becomes

$$\frac{1}{\pm\sqrt{(t^2 - 1)(k^2t^2 - 1)}}, \tag{4.12}$$

where, as before, the sign cannot be chosen arbitrarily. In fact, writing $(t^2 - 1)(1 - k^2t^2)$ in the form

$$-k^2(t - 1)(t + 1)\left(t - \frac{1}{k}\right)\left(t + \frac{1}{k}\right) = h(t),$$

we observe that in going from the interval $1 < x < 1/k$ to the interval $1/k < x < +\infty$, the change in Arg $h(t)$ equals $-\pi$, i.e., the change in Arg $[t - (1/k)]$. Therefore the change in Arg $\sqrt{h(t)}$ is $-\pi/2$, so that Arg $\sqrt{h(t)}$ becomes

$$-\frac{\pi}{2} + \text{Arg}\,\sqrt{g(t)} = -\pi + 2k\pi$$

[cf. (4.11)]. Thus we see that of the two values $\pm\sqrt{(t^2-1)(k^2t^2-1)}$ in (4.12), the correct choice is $-\sqrt{(t^2-1)(k^2t^2-1)}$. It follows that

$$\begin{aligned} w = f(x) &= \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\ &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} + i\int_1^{1/k} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} \\ &\quad - \int_{1/k}^x \frac{dt}{\sqrt{(t^2-1)(k^2t^2-1)}} \\ &= K + iK' - \int_{1/k}^x \frac{dt}{\sqrt{(t^2-1)(k^2t^2-1)}}. \end{aligned}$$

As x increases from $1/k$ to $+\infty$, the last integral on the right becomes

$$\int_{1/k}^{\infty} \frac{dt}{\sqrt{(t^2-1)(k^2t^2-1)}} = \int_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}} = K,$$

where we make the change of variables $t = 1/k\tau$. Correspondingly, the image point $w = f(x)$ describes the line segment parallel to the real axis going from the point $K + iK'$ to the point iK'.

In just the same way, we see that as x moves along the real axis first from 0 to -1, then from -1 to $-1/k$ and finally from $-1/k$ to $-\infty$, the point $w = f(x)$ first describes the segment from 0 to $-K$, then the segment from $-K$ to $-K + iK'$, and finally the segment from $-K + iK'$ to iK'. Thus $w = f(x)$ is a one-to-one mapping of the real axis onto the rectangular contour with vertices $-K$, K, $K + iK'$ and $-K + iK'$, where

$$\begin{aligned} K &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \\ K' &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} \qquad (k'^2 = 1 - k^2). \end{aligned}$$

It follows from Theorem 4.8, Corollary 2 that $w = f(z)$ is univalent in the upper half-plane, which it maps *conformally* (see Theorem 4.1, Corollary 3) onto the indicated rectangle.

Now consider the expression

$$\lambda(k) = \frac{K'}{2K} = \frac{\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}}{\displaystyle 2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}},$$

which is the ratio of the altitude of the rectangle to its base. As the parameter k $(0 < k < 1)$ increases from 0 to 1, the denominator increases from

$$2\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \pi$$

to ∞. At the same time, $k' = \sqrt{1 - k^2}$ decreases from 1 to 0, so that the numerator decreases from ∞ to $\pi/2$. Therefore, as k increases from 0 to 1, the ratio $\lambda(k) = K'/2K$ decreases continuously from ∞ to 0. It follows that given any rectangle with base $2a$ and altitude b, we can find a unique value of k $(0 < k < 1)$ such that

$$\frac{b}{2a} = \frac{\displaystyle\int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k'^2t^2)}}}{\displaystyle\int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}} = \frac{K'}{2K}.$$

Then the function

$$\int_0^z \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}$$

corresponding to this value of k maps the upper half-plane conformally onto a rectangle similar to the given rectangle. To obtain a mapping onto the given rectangle, we need only introduce an extra factor $\mu = 2a/2K = b/K'$, writing

$$w = f(z) = \mu \int_0^z \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}. \tag{4.13}$$

Thus (4.13) maps the upper half-plane onto the rectangle with vertices

$$-\mu K = -a, \quad \mu K = a, \quad \mu K + i\mu K' = a + ib, \quad -\mu K + i\mu K' = -a + ib,$$

and hence onto any given rectangle, provided the parameters k and μ are suitably chosen.

20. The Schwarz-Christoffel Transformation

The mapping (4.7) studied in the preceding section is a special case of a more general mapping known as the *Schwarz-Christoffel transformation*, defined by the integral

$$w = f(z) = C \int_0^z (t - a_1)^{\alpha_1 - 1}(t - a_2)^{\alpha_2 - 1} \cdots (t - a_n)^{\alpha_n - 1}\, dt. \tag{4.14}$$

Here C is a positive constant, $a_1, \ldots, a_n$ are distinct real numbers (assumed to be arranged in increasing order $a_1 < \cdots < a_n$), and the exponents $\alpha_1 - 1, \ldots, \alpha_n - 1$ are also real but not necessarily distinct. For example, (4.14) reduces to the elliptic integral (4.7) if we make the following choice of constants

$$C = \frac{1}{k}, \quad a_1 = -\frac{1}{k}, \quad a_2 = -1, \quad a_3 = 1, \quad a_4 = \frac{1}{k}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}.$$

To guarantee convergence of the integral (4.14), we require that

$$\alpha_k - 1 > -1 \quad \text{or} \quad \alpha_k > 0 \qquad (k = 1, \ldots, n), \tag{4.15}$$

and

$$\alpha_1 + \cdots + \alpha_n - n < -1, \tag{4.16}$$

where (4.16) stems from the fact that the integrand

$$\beta(t) = (t - a_1)^{\alpha_1 - 1} \cdots (t - a_n)^{\alpha_n - 1} \tag{4.17}$$

behaves like $t^{\alpha_1 + \cdots + \alpha_n - n}$ for large t.

Of the various single-valued analytic branches of the function $\beta(t)$ which can be defined in the upper half-plane, we now choose the branch such that the argument of every factor $(t - a_1)^{\alpha_1 - 1}, \ldots, (t - a_n)^{\alpha_n - 1}$ equals π when all the differences $t - a_1, \ldots, t - a_n$ are negative, i.e., when t belongs to the interval $(-\infty, a_1)$. This corresponds to choosing the value

$$(\alpha_1 - 1)\pi + \cdots + (\alpha_n - 1)\pi \tag{4.18}$$

for the argument of $\beta(t)$ when $t \in (-\infty, a_1)$.[13] Then in the other intervals $(a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, +\infty)$, we assign $\operatorname{Arg} \beta(t)$ values which are consistent with this choice, noting that the only factor in (4.17) whose argument changes when we go from the interval (a_{k-1}, a_k) to the interval (a_k, a_{k+1}) is $t - a_k$, giving rise to a change of $-\pi$ in $\operatorname{Arg}(t - a_k)$ and a change of $-(\alpha_k - 1)\pi$ in $\operatorname{Arg} \beta(t)$ itself (cf. footnote 12, p. 125). Since $\operatorname{Arg} \beta(t)$ has the value (4.17) in the interval $(-\infty, a_1)$, it has the value $(\alpha_2 - 1)\pi + \cdots + (\alpha_n - 1)\pi$ in the interval (a_1, a_2), the value $(\alpha_3 - 1)\pi + \cdots + (\alpha_n - 1)\pi$ in the interval (a_2, a_3), and so on. In general, we have

$$\operatorname{Arg} \beta(t) = (\alpha_k - 1)\pi + \cdots + (\alpha_n - 1)\pi \quad \text{if} \quad t \in (a_{k-1}, a_k), \tag{4.19}$$

where $k = 2, \ldots, n$. Finally, to the right of all the points $a_1, \ldots, a_n$, i.e., in the interval $(a_n, +\infty)$, $\operatorname{Arg} \beta(t)$ has the value 0.[14]

Setting

$$w_k = C \int_0^{a_k} (t - a_1)^{\alpha_1 - 1} \cdots (t - a_n)^{\alpha_n - 1}\, dt \qquad (k = 0, 1, \ldots, n + 1), \tag{4.20}$$

we find that (4.14) can be written in the form

$$\begin{aligned} w = f(z) &= w_{k-1} + C \int_{a_{k-1}}^{z} \beta(t)\, dt \\ &= w_{k-1} + C \exp\left[(\alpha_k - 1)\pi + \cdots + (\alpha_n - 1)\pi\right] \int_{a_{k-1}}^{z} |\beta(t)|\, dt \\ &\qquad \text{if} \quad z \in (a_{k-1}, a_k). \end{aligned}$$

---

[13] In (4.18), (4.19), etc., an extra term $+2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) is always understood, since $\operatorname{Arg} z$ is only defined to within an integral multiple of 2π.

[14] Formula (4.19) also holds in the intervals $(-\infty, a_1)$ and $(a_n, +\infty)$, if we introduce extra constants $a_0 = -\infty$, $a_{n+1} = +\infty$ and let k range from 1 to $n + 1$, regarding the right-hand side as zero if $k = n + 1$. Similarly, in (4.20) we allow the upper limit of integration to be any of the points $-\infty = a_0, a_1, \ldots, a_n, a_{n+1} = +\infty$. It should be noted that $w_0 = w_{n+1}$ (why?).

The last term on the right has the same argument for all $z \in (a_{k-1}, a_k)$, but its absolute value increases continuously from 0 to

$$C \int_{a_{k-1}}^{a_k} |\lambda(t)|\, dt = l_k \tag{4.21}$$

as $z = x$ increases from a_{k-1} to a_k. Therefore, as z traverses the interval (a_{k-1}, a_k) from left to right, the image point $w = f(z)$ traverses the line segment Δ_{k-1} of length l_k, with initial point w_{k-1} and final point w_k, making an angle of $(\alpha_k - 1)\pi + \cdots + (\alpha_n - 1)\pi$ radians with the positive real axis. This applies equally well to the line segment Δ_0, with initial point

$$w_0 = C \int_0^{\infty} (t - a_1)^{\alpha_1 - 1} \cdots (t - a_n)^{\alpha_n - 1}\, dt$$

and final point w_1, and the line segment Δ_n, with initial point w_n and final point $w_{n+1} = w_0$. It follows that the function (4.14) maps the real axis onto a closed polygonal curve Λ with segments $\Delta_0, \Delta_1, \ldots, \Delta_n$ and vertices $w_0, w_1, \ldots, w_n, w_{n+1} = w_0$. It cannot be asserted that this mapping is one-to-one, since the polygonal curve may have self-intersections, i.e., the segments Δ_k may have points in common other than the common end points of consecutive segments. However, if we make the additional assumption that there are no self-intersections [as, for example, in the case of the elliptic integral (4.7)], then Λ will be a closed rectifiable Jordan curve, bounding a polygon with vertices $w_0, w_1, \ldots, w_n$. Therefore, in this case, we can apply Theorem 4.8, Corollary 2, concluding that the Schwarz-Christoffel transformation (4.14) is univalent in the upper half-plane, which it maps conformally onto the indicated polygon.

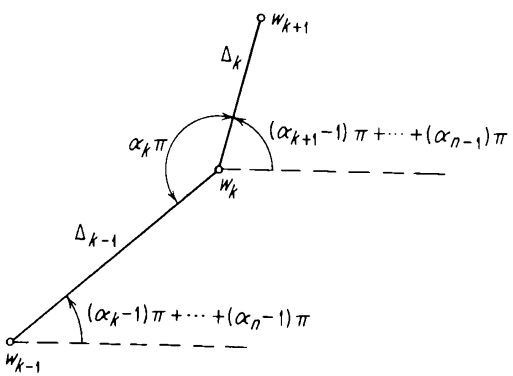

FIGURE 4.2

According to the construction shown in Figure 4.2, the interior angle of the polygon at the vertex w_k $(0 < k < n + 1)$ equals $\alpha_k\pi$. Although all the angles in the figure are only defined to within integral multiples of 2π, it is easy to see that $\alpha_k\pi$ gives the interior angle without any correction term, i.e., that the inequality $\alpha_k < 2$ holds as well as the inequality $\alpha_k > 0$ assumed in (4.15). In fact, we can write

$$\beta(t) = [(t - a_1)^{\alpha_1 - 1} \cdots (t - a_{k-1})^{\alpha_{k-1} - 1} (t - a_{k+1})^{\alpha_{k+1} - 1} \cdots (t - a_n)^{\alpha_n - 1}] \times (t - a_k)^{\alpha_k - 1},$$

where the expression in brackets is analytic and nonvanishing in a neighborhood of the point a_k, and hence has a power series expansion of the form

$$A_0^{(k)} + A_1^{(k)}(t - a_k) + \cdots \qquad (A_0^{(k)} \neq 0)$$

which converges uniformly in a neighborhood of a_k. But then

$$w = w_k + C\int_{a_k}^{z} \beta(t)\,dt = w_k + C\int_{a_k}^{z} [A_0^{(k)} + A_1^{(k)}(t - a_k) + \cdots](t - a_k)^{\alpha_k - 1}\,dt$$
$$= w_k + C\frac{A_0^{(k)}}{\alpha_k}(z - a_k)^{\alpha_k} + C\frac{A_1^{(k)}}{\alpha_k + 1}(z - a_k)^{\alpha_k + 1} + \cdots,$$

which implies

$$w - w_k = C\frac{A_0^{(k)}}{\alpha_k}(z - a_k)^{\alpha_k}\left[1 + \frac{\alpha_k}{\alpha_k + 1}\frac{A_1^{(k)}}{A_0^{(k)}}(z - a_k) + \cdots\right] \qquad (A_0^{(k)} \neq 0).$$

It follows that as z approaches a_k along a ray drawn from a_k making an angle θ with the positive real axis, w approaches w_k along some curve whose tangent at the point w_k has an inclination equal to

$$\lim_{w \to w_k} \operatorname{Arg}(w - w_k) = \operatorname{Arg} A_0^{(k)} + \alpha_k\theta,$$

where we use the fact that $C > 0$, $\alpha_k > 0$. Thus, as θ varies from 0 to π along a sufficiently small semicircle with center a_k, the image point $w = f(z)$ varies continuously along some Jordan curve joining a point on the segment Δ_k to a point on the segment Δ_{k-1}, always staying inside the interior angle with vertex at w_k, while $\operatorname{Arg} w$ increases from $\operatorname{Arg} A_0^{(k)}$ to $\operatorname{Arg} A_0^{(k)} + \alpha_k\pi$, without winding around the point $w = 0$ (here we use the univalence and conformality of the mapping). Therefore $\alpha_k\pi$ is just the size of some interior angle of the polygon, and hence must be less than 2π. In other words, instead of the condition (4.15), we have the stronger condition

$$0 < \alpha_k < 2\pi \qquad (k = 1, \ldots, n). \tag{4.22}$$

At the vertex $w_0 = w_{n+1}$, the interior angle of the polygon equals

$$[(n - 1) - (\alpha_1 + \cdots + \alpha_n)]\pi, \tag{4.23}$$

as shown by Figure 4.3. From (4.16) we can only deduce that this quantity is positive. However, our polygon has a total of $n + 1$ interior angles, whose sum must equal $[(n + 1) - 2]\pi = (n - 1)\pi$. Therefore (4.23) is precisely the size of the $(n + 1)$st interior angle, and hence

$$[(n - 1) - (\alpha_1 + \cdots + \alpha_n)]\pi < 2\pi,$$

or

$$\alpha_1 + \cdots + \alpha_n > n - 3.$$

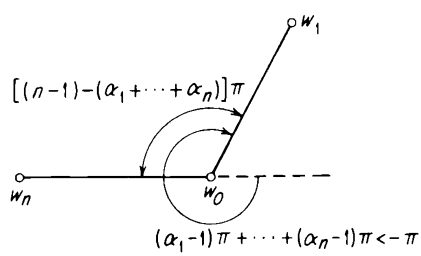

FIGURE 4.3

In other words, instead of the condition (4.16), we have the stronger condition

$$n - 3 < \alpha_1 + \cdots + \alpha_n < n - 1. \tag{4.24}$$

The conditions (4.23) and (4.24) are necessary, but not sufficient, for the Schwarz-Christoffel transformation (4.14) to be a one-to-one conformal mapping of the upper half-plane onto a bounded polygon, with vertices $w_0, w_1, \ldots, w_n$, and interior angles

$$[(n - 1) - (\alpha_1 + \cdots + \alpha_n)]\pi, \alpha_1\pi, \ldots, \alpha_n\pi.$$

In the special case where

$$\alpha_1 + \cdots + \alpha_n = n - 2,$$

we find that the angle with vertex w_0 equals π. Geometrically, this means that w_0 is not really a vertex, but just an interior point of the side with vertices w_n and w_1 (cf. Figure 4.3). Then (4.14) maps the upper half-plane onto an n-gon rather than an $(n + 1)$-gon. This occurs in the case of the elliptic integral (4.7), where

$$n = 4, \qquad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4 \cdot \tfrac{1}{2} = n - 2,$$

and in the following example:

Example 1. We now use the Schwarz-Christoffel transformation to map the upper half-plane onto a triangle with angles $\alpha_1\pi$, $\alpha_2\pi$, $\alpha_3\pi$ ($\alpha_k > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$), where the side opposite the angle $\alpha_1\pi$ has length l. This problem can be solved in two ways. First we use formula (4.14) with $n = 3$, choosing three arbitrary real points a_1, a_2, a_3, which are mapped into the vertices of the triangle (in this approach, $\alpha_1 + \alpha_2 + \alpha_3 = n - 2$). Thus, for example, suppose that

$$a_1 = -1, \quad a_2 = 0, \quad a_3 = 1.$$

Then (4.14) gives

$$w = C \int_0^z (t + 1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (t - 1)^{\alpha_3 - 1}\, dt, \tag{4.25}$$

and we need only determine the factor $C > 0$. But since l must equal

$$C \int_0^1 |(t + 1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (t - 1)^{\alpha_3 - 1}|\, dt = C \int_0^1 (t + 1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (1 - t)^{\alpha_3 - 1}\, dt$$

[cf. (4.21)], we have

$$C = \frac{l}{\displaystyle\int_0^1 (t + 1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (1 - t)^{\alpha_3 - 1}\, dt}.$$

Therefore (4.25) becomes

$$w = l \frac{\displaystyle\int_0^z (t + 1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (t - 1)^{\alpha_3 - 1}\, dt}{\displaystyle\int_0^1 (t + 1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (1 - t)^{\alpha_3 - 1}\, dt}. \tag{4.26}$$

This function actually solves the problem, since it is one-to-one on the real axis, which it maps onto a triangle with the given vertex angles and side length (note that a triangle can have no self-intersections). In particular, (4.26) reduces to

$$w = l\frac{\displaystyle\int_0^z \frac{dt}{\sqrt[3]{t^2(t^2-1)^2}}}{\displaystyle\int_0^1 \frac{dt}{\sqrt[3]{t^2(1-t^2)^2}}}$$

if $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ (corresponding to an equilateral triangle with side length l), and to

$$w = l\frac{\displaystyle\int_0^z \frac{dt}{\sqrt{t}\sqrt[4]{(t^2-1)^3}}}{\displaystyle\int_0^1 \frac{dt}{\sqrt{t}\sqrt[4]{(1-t^2)^3}}}$$

if $\alpha_2 = \frac{1}{2}$, $\alpha_1 = \alpha_3 = \frac{1}{4}$ (corresponding to an isosceles right triangle with hypotenuse l).

Example 2. Another approach to the problem of mapping the upper half-plane onto a triangle is based on the use of formula (4.14) with $n = 2$. This time we choose *two* arbitrary real points a_1 and a_2, which are to be mapped into two vertices of the triangle, with angles $\alpha_1\pi$ and $\alpha_2\pi$. Since $\alpha_1 + \alpha_2$ cannot equal $n - 2 = 0$, the point at infinity must go into the third vertex of the triangle, with angle

$$[(2-1) - (\alpha_1 + \alpha_2)]\pi = \pi - \alpha_1\pi - \alpha_2\pi = \alpha_3\pi.$$

Thus, for example, suppose that

$$a_1 = 0, \qquad a_2 = -1.$$

Then (4.14) gives

$$w = C\int_0^z t^{\alpha_1 - 1}(t-1)^{\alpha_2 - 1}\,dt,$$

and to determine C we have the equation

$$C\int_1^\infty |t^{\alpha_1 - 1}(t-1)^{\alpha_2 - 1}|\,dt = l,$$

which implies

$$C = \frac{l}{\displaystyle\int_1^\infty t^{\alpha_1 - 1}(t-1)^{\alpha_2 - 1}\,dt}.$$

It follows that

$$w = l\frac{\displaystyle\int_0^z t^{\alpha_1 - 1}(t-1)^{\alpha_2 - 1}\,dt}{\displaystyle\int_1^\infty t^{\alpha_1 - 1}(t-1)^{\alpha_2 - 1}\,dt}, \tag{4.27}$$

which obviously differs from (4.26). In particular, in the case of an equilateral triangle with side l, (4.27) reduces to

$$w = l \frac{\int_0^z \frac{dt}{\sqrt[3]{t^2(t-1)^2}}}{\int_1^\infty \frac{dt}{\sqrt[3]{t^2(t-1)^2}}}.$$

Example 3. In the general case, the upper half-plane is to be mapped onto an n-gon ($n \geqslant 4$) with interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$ and side lengths $l_1, \ldots, l_n$, where, as in Figure 4.4, we associate the angles $\alpha_{k-1}\pi$ and $\alpha_k\pi$ with the side l_k ($k = 2, \ldots, n$), and the angles $\alpha_n\pi$ and $\alpha_1\pi$ with the side l_1. Suppose we choose three arbitrary points a_1, a_2 and a_3 of the real axis, which are to go into the vertices w_1, w_2 and w_3, with angles $\alpha_1\pi$, $\alpha_2\pi$ and $\alpha_3\pi$. Then we must still determine $n - 3$ points $a_4, \ldots, a_n$ of the real axis, as well as the constant $C > 0$, which makes $n - 2$ unknowns in all. At first glance, it appears that there are too many equations available for determining these unknowns, i.e., the n equations expressing the side lengths in terms of appropriate integrals. However, it is immediately apparent from Figure 4.4 that once the angles of an n-gon are known, then specifying just $n - 2$ sides $l_1, \ldots, l_{n-2}$ uniquely determines the remaining two sides l_{n-1} and l_n.[15] In other words, there are actually only $n - 2$ independent equations of the form

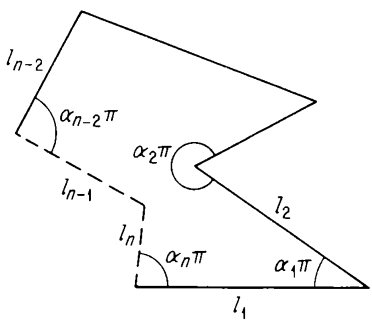

FIGURE 4.4

$$C\int_{a_{k-1}}^{a_k} |(t-a_1)^{\alpha_1-1}\cdots(t-a_n)^{\alpha_n-1}|\,dt = l_k \qquad (k = 1, 2, \ldots, n-2),$$

for determining the $n - 2$ unknowns $a_4, \ldots, a_n$, C. However, the proof of the fact that the constants in (4.14) can always be chosen in such a way as to map the upper half-plane onto an arbitrarily given n-gon requires the use of results more powerful than any proved in the present chapter. The necessary theory will be developed in Volume III of this course, where we shall once again consider the Schwarz-Christoffel transformation, carrying the subject further than in this section, and permitting interior angles equal to 0 or 2π, sides of infinite length, etc.

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[15] Obviously, specifying $n - 1$ angles of an n-gon uniquely determines the remaining angle.

*21. Sufficient Conditions for Univalent Mapping onto a Half-Plane

The following result is in the same spirit as Theorem 4.9, but involves a condition of an entirely different kind:

THEOREM 4.10. *Given a closed rectifiable Jordan curve* γ, *suppose* $w = f(z)$ *is analytic on* $\overline{I(\gamma)}$, *except at a finite number of points* $z_0, z_1, \ldots, z_n$ *belonging to* γ, *at one of which, namely* z_0, $f(z)$ *becomes infinite, so that*

$$\lim_{\substack{z \to z_0 \\ z \in \overline{I(\gamma)}}} f(z) = \infty.$$

Suppose γ *has left and right-hand tangents*[16] *at* z_0, *forming an angle of* $\alpha\pi$ *radians* $(0 < \alpha \leqslant 2)$ *with its vertex at* z_0, *and suppose the condition*

$$\lim_{\substack{z \to z_0 \\ z \in \overline{I(\gamma)}}} (z - z_0)^{\lambda} f(z) = c \qquad (c \neq 0,\ c \neq \infty) \tag{4.28}$$

is satisfied for constants c *and* λ, *where* $0 < \lambda < 3/\alpha$. *Finally, suppose* $f(z)$ *is wide-sense continuous on* $\overline{I(\gamma)}$ *and one-to-one on* γ, *which it maps onto the real axis* Γ *in the* w*-plane. Then* $f(z)$ *is univalent on* $I(\gamma)$, *which it maps onto one of the half-planes with boundary* Γ, *i.e., the half-plane* Π_1 *on the left of an observer moving with the point* $w = f(z)$ *as* z *traverses* γ *once in the positive direction.*

Proof. Before proceeding to the proof proper, we say a few words clarifying the meaning of the condition (4.28). Suppose $\alpha = 1$, so that the curve γ has a tangent at the point z_0. Then the positive constant λ must be less than 3, which is consistent with $f(z)$ having a pole at z_0 of order less than 3, i.e., with either of the conditions

$$\lim_{\substack{z \to z_0 \\ z \in \overline{I(\gamma)}}} (z - z_0) f(z) = c_1 \neq 0 \qquad (\lambda = 1),$$

$$\lim_{\substack{z \to z_0 \\ z \in \overline{I(\gamma)}}} (z - z_0)^2 f(z) = c_2 \neq 0 \qquad (\lambda = 2).$$

However the theorem fails to hold if the pole is of order 3, as we have already seen in the case of the mapping

$$f(z) = i\left(\frac{z - 1}{z + 1}\right)^3,$$

considered in the example on p. 123. Thus, if $\alpha = 1$, we cannot replace the inequality in the condition $\lambda < 3$ by an equality.

Turning to the proof itself, let $\sigma_{k,\rho}$, $\gamma_{k,\rho}$, $D_{k,\rho}$, γ_ρ, $\Sigma_{k,\rho} = f(\sigma_{k,\rho})$, $\Gamma_{k,\rho} = f(\gamma_{k,\rho})$ and $\Gamma_\rho = f(\gamma_\rho)$ have exactly the same meaning as in

[16] See Problem I.8.1.

the proof of Theorem 4.8, except that now k ranges from 0 to n, since there are $n + 1$ points $z_0, z_1, \ldots, z_n$, instead of n points $z_1, \ldots, z_n$. Let w' be an arbitrary point of the half-plane Π_1, and choose disjoint neighborhoods $\mathcal{N}(w_k)$ so small that

$$w' \notin \mathcal{N}(w_k), \quad \Sigma_{k,\rho} \subset \mathcal{N}(w_k), \quad \Gamma_{k,\rho} \subset \mathcal{N}(w_k) \qquad (k = 0, 1, \ldots, n)$$

(note that $w_0 = \infty$). Then, just as before, we have

$$\operatorname*{Var}_{z \in \sigma_{k,\rho}} \operatorname{Arg} [f(z) - w'] = \operatorname*{Var}_{z \in \gamma_{k,\rho}} \operatorname{Arg} [f(z) - w']$$

if $k \neq 0$, so that adding all the changes except those caused by traversing the arcs $\sigma_{0,\rho}$ and $\gamma_{0,\rho}$, we find that

$$\operatorname*{Var}_{z \in \gamma_\rho - \sigma_{0,\rho}} \operatorname{Arg} [f(z) - w'] = \operatorname*{Var}_{z \in \gamma - \gamma_{0,\rho}} \operatorname{Arg} [f(z) - w']. \tag{4.29}$$

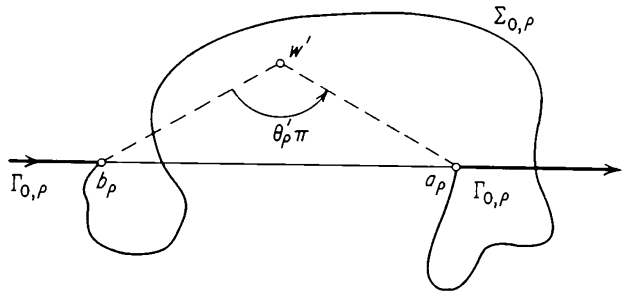

FIGURE 4.5

Let a_ρ and b_ρ denote the common initial and final points of the arcs $\Gamma_{0,\rho}$ and $\Sigma_{0,\rho}$ (traversed in the positive direction). Clearly, a_ρ and b_ρ are points of the real axis. Correspondingly, $\Gamma_{0,\rho}$ is the infinite segment of the real axis with initial point a_ρ and final point b_ρ, while $\Gamma - \Gamma_{0,\rho}$ is the finite segment with initial point b_ρ and final point a_ρ (see Figure 4.5).[17] Therefore, if w traverses the segment $[b_\rho, a_\rho]$ in such a way that the vector $w - w'$ rotates around w' in the positive direction,[18] we have

$$\operatorname*{Var}_{z \in \gamma - \gamma_{0,\rho}} \operatorname{Arg} [f(z) - w'] = \operatorname*{Var}_{w \in [b_\rho, a_\rho]} \operatorname{Arg} (w - w') = 0'_\rho \pi, \tag{4.30}$$

where $0'_\rho \pi$ $(0 < 0' < 1)$ is just the angle subtended at the point w' by the segment $[b_\rho, a_\rho]$. In fact, since

$$\operatorname*{Var}_{w \in [b_\rho, a_\rho]} \operatorname{Arg} (w - w') = \operatorname{Arg} (a_\rho - w') - \operatorname{Arg} (b_\rho - w')$$

$$= \operatorname{Arg} \frac{b_\rho - w'}{a_\rho - w'},$$

[17] In Figure 4.5 it is not yet known that the mapping $w = f(z)$ is univalent, and hence we allow both the upper and lower half-planes to contain points of $\Sigma_{0,\rho}$.

[18] Recall that the point w' is seen on the left by an observer moving with w from b_ρ to a_ρ.

we have

$$\theta'_\rho \pi = \arg \frac{b_\rho - w'}{a_\rho - w'}.$$

It follows from (4.29) and (4.30) that

$$\begin{aligned} &\operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg} [f(z) - w'] \\ &\quad = \operatorname*{Var}_{z \in \gamma_\rho} \operatorname{Arg} [f(z) - w'] - \operatorname*{Var}_{z \in \gamma_\rho - \sigma_{0,\rho}} \operatorname{Arg} [f(z) - w'] \\ &\quad = \operatorname*{Var}_{w \in \Gamma_\rho} \operatorname{Arg} (w - w') - \operatorname*{Var}_{z \in \gamma - \gamma_{0,\rho}} \operatorname{Arg} [f(z) - w'] \\ &\quad = 2m'\pi - \theta'_\rho \pi, \end{aligned} \tag{4.31}$$

where m' is an integer. On the other hand, in a neighborhood of the point z_0, the function $f(z)$ can be written in the form

$$f(z) = \frac{c[1 + \varepsilon(z)]}{(z - z_0)^\lambda},$$

where $\varepsilon(z) \to 0$ as $z \to z_0$. Therefore

$$f(z) - w' = \frac{c}{(z - z_0)^\lambda} \left[1 + \varepsilon(z) - \frac{w'(z - z_0)^\lambda}{c}\right] = \frac{c}{(z - z_0)^\lambda} [1 + \eta(z)],$$

where $\eta(z) \to 0$ as $z \to z_0$, which implies

$$\operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg} [f(z) - w'] = -\lambda \operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg} (z - z_0) + \operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg} [1 + \eta(z)].$$

Since the tangents to the curve γ at the point z_0 form an angle of $\alpha\pi$ radians, all points of the arc $\gamma_{0,\rho}$ lie inside two angles with bisectors along the tangents, where the angles can be made arbitrarily small by choosing ρ sufficiently small (cf. Lemma 3, p. I.311). Thus the angle $\alpha_\rho \pi$ subtended by the arc $\sigma_{0,\rho}$ at the point z_0 can be made arbitrarily close to $\alpha\pi$ by choosing ρ sufficiently small. Noting that z traverses $\sigma_{0,\rho}$ in such a way that z_0 is on the *right* of an observer moving with z, we have

$$\operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg} (z - z_0) = -\alpha_\rho \pi, \tag{4.32}$$

and hence

$$\operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg} [f(z) - w'] = \lambda \alpha_\rho \pi + \beta_\rho \pi, \tag{4.33}$$

where

$$\lim_{\rho \to 0} \alpha_\rho = \alpha, \qquad \lim_{\rho \to 0} \beta_\rho = \lim_{\rho \to 0} \frac{1}{\pi} \operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg} [1 + \eta(z)] = 0.$$

Comparing (4.31) and (4.33), we find that

$$2m' = \theta'_\rho + \lambda \alpha_\rho + \beta_\rho. \tag{4.34}$$

As $\rho \to 0$, the points a_ρ and b_ρ approach $+\infty$ and $-\infty$, respectively, so that the angle $\theta'_\rho\pi$ approaches π. Therefore the right-hand side of (4.34) approaches $1 + \lambda\alpha$, where, by hypothesis,

$$1 < 1 + \lambda\alpha < 4.$$

It follows that $m' = 1$ (and hence $\lambda = 1/\alpha$) for all sufficiently small ρ. But then (4.31) implies

$$\operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg}[f(z) - w'] = 2\pi - \theta'_\rho\pi,$$

so that

$$\begin{aligned}\operatorname*{Var}_{z \in \gamma_\rho} &\operatorname{Arg}[f(z) - w'] \\ &= \operatorname*{Var}_{z \in \gamma_\rho - \sigma_{0,\rho}} \operatorname{Arg}[f(z) - w'] + \operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg}[f(z) - w'] \\ &= \theta'_\rho\pi + (2\pi - \theta'_\rho\pi) = 2\pi.\end{aligned}$$

In other words, $f(z)$ takes every value $w' \in \Pi_1$ once at a point of $I(\gamma_\rho)$, for all sufficiently small ρ.

Finally, let Π_2 be the other half-plane with boundary Γ, i.e., the half-plane on the *right* of an observer moving with the point $w = f(z)$ as z traverses γ once in the positive direction, and let w'' be an arbitrary point of Π_2. Then (4.31) becomes

$$\operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg}[f(z) - w''] = 2m''\pi + \theta''_\rho\pi,$$

where m'' is an integer, and $\theta''_\rho\pi$ $(0 < \theta'' < 1)$ is just the angle subtended at the point w'' by the segment $[b_\rho, a_\rho]$. The same reasoning as before shows that $m'' = 0$, so that

$$\begin{aligned}\operatorname*{Var}_{z \in \gamma_\rho} &\operatorname{Arg}[f(z) - w''] \\ &= \operatorname*{Var}_{z \in \gamma_\rho - \sigma_{0,\rho}} \operatorname{Arg}[f(z) - w''] + \operatorname*{Var}_{z \in \sigma_{0,\rho}} \operatorname{Arg}[f(z) - w''] \\ &= -\theta''_\rho\pi + \theta''_\rho\pi = 0.\end{aligned}$$

It follows that $f(z)$ takes no value $w'' \in \Pi_2$ (in fact, no value $w'' \in \Pi_2 \cup \Gamma$) at a point of $I(\gamma_\rho)$, for all sufficiently small ρ. The proof now follows at once, in the usual way.

PROBLEMS

4.1. In connection with Theorem 4.4, give an example of a sequence of univalent functions which converges uniformly on some domain to a constant.

4.2. Show that the word "univalent" can be replaced by the word "homeomorphic" in the statements of Theorems 4.5–4.10.

Hint. Recall Theorem I.6.1.

4.3. Verify that the mapping

$$f(z) = i\left(\frac{z-1}{z+1}\right)^3,$$

considered in the example on p. 123, fails to meet the requirements of Theorem 4.9.

4.4. Generalize Theorem 4.9 one step further, by considering the case where γ, as well as its image $\Gamma = f(\gamma)$, is unbounded.

4.5. Generalize Theorem 4.5 in the direction of allowing $f(z)$ to become infinite *inside* γ. Specifically, prove the following theorem: Given a closed rectifiable Jordan curve γ, suppose $f(z)$ is analytic on $\overline{I(\gamma)}$ except at a point $z_0 \in I(\gamma)$ where it has a simple pole, and suppose $f(z)$ is one-to-one on γ. Then $f(z)$ is univalent on $I(\gamma)$, which it maps onto $E(\Gamma)$, where $\Gamma = f(\gamma)$.

Hint. Use the argument principle.

4.6. Give an example of a function which is univalent on a domain but not one-to-one on the boundary of G.

Hint. Cf. Remark 1, p. I.201.

4.7. Is the function $f(z) = \sin z - z$ univalent in any neighborhood of the origin? Is it univalent in any neighborhood of the point $z = 1$?

4.8. Find the image of the first quadrant Q: Re $z > 0$, Im $z > 0$ under the mapping

$$w = \int_0^z \frac{dt}{\sqrt{1-t^2}}.$$

Is the mapping univalent on Q?

4.9. Map the upper half-plane Im $z > 0$ onto a rhombus with side length l and obtuse angle $\alpha\pi$.

4.10. Derive the formula for mapping the unit disk $|z| < 1$ onto the interior of a polygon.

Hint. Make a preliminary Möbius transformation mapping the upper half-plane onto the unit disk.

4.11. Prove that the function

$$w = \int_0^z \frac{dt}{\sqrt{1-t^4}}$$

maps the unit disk onto a square. What is the side length of this square?

4.12. Prove that the function

$$w = \int_0^z \frac{dt}{(1-t^2)^{2/n}}$$

maps the unit disk onto a regular n-gon. What is the side length of this n-gon?

4.13. Onto what domain is the unit disk mapped by the function

$$w = \int_0^z \frac{(1 + t^n)^\lambda}{(1 - t^2)^{\frac{2}{n}+\lambda}}\, dt,$$

where

$$-1 < \lambda < 1 - \frac{2}{n}?$$

Hint. The domain is star-shaped.

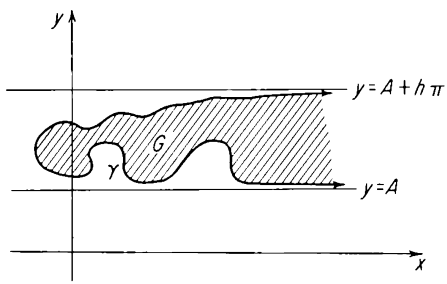

FIGURE 4.6

4.14. Prove the following theorem, of the same type as Theorem 4.10: Let G be a "curvilinear half-strip," i.e., an infinite domain like the one shown in Figure 4.6, where the lines $y = A$ and $y = A + h\pi$ $(0 < h \leqslant 2\pi)$ are asymptotes for γ, the boundary of G, and every finite arc of γ is rectifiable. Suppose $f(z)$ is analytic on $\bar{G}$ except at $z = \infty$, where $f(z)$ becomes infinite, i.e.,

$$\lim_{\substack{z \to \infty \\ z \in \bar{G}}} f(z) = \infty,$$

and suppose the condition

$$\lim_{\substack{z \to \infty \\ z \in \bar{G}}} z^\varkappa e^{-\lambda z} f(z) = c \qquad (c \neq 0, c \neq \infty)$$

is satisfied for constants c, $\varkappa$ and λ, where $-\infty < \varkappa < \infty$, $0 < \lambda < 3/h$. Moreover, suppose $f(z)$ is wide-sense continuous on $\bar{G}$ and one-to-one on γ, which it maps onto the real axis Γ in the w-plane. Then $f(z)$ is univalent on G, which it maps onto one of the half-planes with boundary Γ.

Hint. Apply Theorem 4.10 to the image of G under the mapping $\zeta = z^\mu e^{-z}$, where μ is an arbitrary real number (a preliminary shift of G may be required.)

Part 2

HARMONIC AND SUBHARMONIC FUNCTIONS

5

BASIC PROPERTIES OF HARMONIC FUNCTIONS

22. Laplace's Equation. Conjugate Harmonic Functions

A function $u(x, y)$ of two real variables is said to be *harmonic* on a domain G if the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial x\, \partial y}, \quad \frac{\partial^2 v}{\partial y^2}$$

exist and are continuous on G, and if at every point of G, $u(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{5.1}$$

known as *Laplace's equation*. Let $u(x, y)$ and $v(x, y)$ be two functions harmonic on a domain G, which satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5.2}$$

at every point of G. Then $u(x, y)$ and $v(x, y)$ are said to be *conjugate harmonic functions* (on G), and each of the functions $u(x, y)$, $v(x, y)$ is said to be the conjugate harmonic function (or simply the *harmonic conjugate*) of the other. There is an intimate connection between harmonic functions and analytic

functions, as shown by

THEOREM 5.1. *A necessary and sufficient condition for a function* $f(z) = u(x, y) + iv(x, y)$ *to be analytic on a domain* G *is that its real part* $u(x, y)$ *and imaginary part* $v(x, y)$ *be conjugate harmonic functions on* G.

Proof. To prove the necessity, suppose $f(z)$ is analytic on G, i.e., differentiable at every point of G. Then, according to Theorem I.7.2, the functions $u(x, y)$ and $v(x, y)$ are differentiable and satisfy the Cauchy-Riemann equations (5.2) at every point of G. The function $f'(z)$ is also analytic on G, being the derivative of a function analytic on G. Since

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x},$$

we see that

$$\frac{\partial u}{\partial x}, \quad -\frac{\partial u}{\partial y} \tag{5.3}$$

and

$$\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} \tag{5.4}$$

are pairs of differentiable functions on G, satisfying the Cauchy-Riemann equations. Writing the first Cauchy-Riemann equation for (5.3) and the second for (5.4), we find that

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right), \qquad \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right),$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Thus each of the functions $u(x, y)$ and $v(x, y)$ satisfies Laplace's equation (5.1). It only remains to show that $u(x, y)$ and $v(x, y)$ have *continuous* second derivatives. But this follows at once from the fact that $f''(z)$ is analytic on G (why?) and can be written in any of the forms

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial x\,\partial y} - i\frac{\partial^2 u}{\partial x\,\partial y}.$$

To prove the sufficiency, we note that if $u(x, y)$ and $v(x, y)$ are conjugate harmonic functions, then, in particular, they have continuous first derivatives on G, and hence are differentiable on G. Since $u(x, y)$ and $v(x, y)$ also satisfy the Cauchy-Riemann equations on G, it follows from Theorem I.7.2 that $f(z) = u(x, y) + iv(x, y)$ is analytic on G.

Example 1. The function

$$f(z) = e^z = e^x(\cos y + i \sin y)$$

is analytic in the whole plane, and hence its real and imaginary parts

$$u(x, y) = e^x \cos y, \qquad v(x, y) = e^x \sin y \tag{5.5}$$

are harmonic in the whole plane. It is easily verified that both functions (5.5) satisfy Laplace's equation.

Example 2. Let G be the whole plane cut along the negative real axis (including the origin). Then every single-valued branch of the multiple-valued function

$$f(z) = \operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z$$

is analytic on G. It follows that

$$u(x, y) = \ln |z| = \ln \sqrt{x^2 + y^2}, \qquad v(x, y) = \arg z + 2k\pi$$

is a pair of conjugate harmonic functions on G, for every $k = 0, \pm 1, \pm 2, \ldots$. If $v(x, y)$ is thought of as a multiple-valued function, then it has infinitely many "single-valued harmonic branches."

More generally, let

$$f(z) = \operatorname{Ln} \frac{z - z_1}{z - z_2},$$

and let G be the plane cut along the segment σ joining z_1 to z_2. Then our harmonic functions become

$$u(x, y) = \ln \frac{|z - z_1|}{|z - z_2|}, \quad v(x, y) = i \arg \frac{z - z_1}{z - z_2} + 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots),$$

where $u(x, y)$ is the logarithm of the ratio of the distance between z and z_1 to the distance between z and z_2, while $v(x, y)$ is the angle subtended by the segment σ at the point z.

Example 3. Let $f(z)$ be single-valued and analytic on an annulus $D: \rho < |z - z_0| < R$. Then $f(z)$ has the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (z \in D),$$

or in polar coordinates,

$$f(z) = \sum_{n=-\infty}^{\infty} \rho_n r^n e^{i(n\theta + \varphi_n)} = \sum_{n=-\infty}^{\infty} \rho_n r^n [\cos(n\theta + \varphi_n) + i \sin(n\theta + \varphi_n)],$$

where we write

$$a_n = \rho_n e^{i\varphi_n}, \qquad z - z_0 = re^{i\theta}.$$

This implies

$$u(r, \theta) = \operatorname{Re} f(z) = \sum_{n=-\infty}^{\infty} \rho_n r^n \cos(n\theta + \varphi_n),$$

$$v(r, \theta) = \operatorname{Im} f(z) = \sum_{n=-\infty}^{\infty} \rho_n r^n \sin(n\theta + \varphi_n).$$

It is customary to write $u(x, y)$ and $v(x, y)$ for the expressions obtained from $u(r, \theta)$ and $v(r, \theta)$ after transforming back to rectangular coordinates.[1] Then

[1] This entails the following slight abuse of notation:

$$u(x, y) = u(r, \theta), \qquad v(x, y) = v(r, \theta).$$

the nth term of the series for $u(x, y)$ turns out to be a homogeneous polynomial of degree n in x and y, called a *harmonic polynomial* (see Problem 5.3). From the standpoint of simplicity, the expressions in polar coordinates are preferable to those in rectangular coordinates.

Example 4. Later on, an important role will be played by the real and imaginary parts of the function

$$f(z) = \frac{\rho e^{i\varphi} + (z - z_0)}{\rho e^{i\varphi} - (z - z_0)},$$

which is analytic on the disk $K: |z - z_0| < \rho$. Writing $z - z_0 = re^{i\theta}$, and denoting the real and imaginary parts of $f(z)$ in polar coordinates by $u(r, \theta)$ and $v(r, \theta)$, as before, we have

$$u(r, \theta) + iv(r, \theta) = \frac{\rho e^{i\varphi} + re^{i\theta}}{\rho e^{i\varphi} - re^{i\theta}} = \frac{(\rho e^{i\varphi} + re^{i\theta})(\rho e^{-i\varphi} - re^{-i\theta})}{(\rho e^{i\varphi} - re^{i\theta})(\rho e^{-i\varphi} - re^{-i\theta})}$$
$$= \frac{\rho^2 - r^2 + 2i\rho r \sin(\theta - \varphi)}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}.$$

The corresponding (single-valued) conjugate harmonic functions on K are

$$u(r, \theta) = \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}, \qquad v(r, \theta) = \frac{2\rho r \sin(\theta - \varphi)}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}.$$

Both expressions have the same denominator

$$(\rho e^{i\varphi} - re^{i\theta})(\rho e^{-i\varphi} - re^{-i\theta}) = |\rho e^{i\varphi} - re^{i\theta}|^2 = |\rho e^{i\varphi} - (z - z_0)|^2,$$

equal to the square of the distance between the point z and the point $z_0 + \rho e^{i\varphi}$ on the circle $|z - z_0| = \rho$.

Next we ask whether every function $u(x, y)$ harmonic on a given domain G can be regarded as the real (or imaginary) part of a function $f(z)$ analytic on G. In other words, can we always find another function $v(x, y)$ harmonic on G which is a harmonic conjugate of $u(x, y)$, i.e., which together with $u(x, y)$ satisfies the Cauchy-Riemann equations (5.2) on G? The answer to this question is in the affirmative, as shown by

THEOREM 5.2. *Given a function $u(x, y)$ harmonic on a simply connected domain G, then, to within an arbitrary real constant, the function*

$$v(x, y) = \int_{(x_0, y_0)}^{(x, y)} \left(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right) \tag{5.6}$$

is the unique harmonic conjugate of $u(x, y)$ on G, where the line integral is evaluated along any rectifiable curve $L \subset G$ joining the fixed point (x_0, y_0) to the variable point (x, y). Similarly, to within an arbitrary purely imaginary constant, the function

$$f(z) = u(x, y) + iv(x, y) = u(x, y) + i \int_{(x_0, y_0)}^{(x, y)} \left(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right) \tag{5.7}$$

is the unique analytic function on G with $u(x, y)$ as its real part.

Proof. Clearly, the proof reduces to finding all the solutions of the system

$$\frac{\partial v}{\partial x} = P(x, y), \qquad \frac{\partial v}{\partial y} = Q(x, y) \tag{5.8}$$

on G, where

$$P(x, y) = -\frac{\partial u}{\partial y}, \qquad Q(x, y) = \frac{\partial u}{\partial x} \tag{5.9}$$

are given functions with continuous first partial derivatives. In terms of the functions (5.9), Laplace's equation for $u(x, y)$ becomes

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

But this is just the condition for

$$v(x, y) = \int_{(x_0, y_0)}^{(x,y)} \{P(x, y)\, dx + Q(x, y)\, dy\} \tag{5.10}$$

to be a solution of (5.8) on G,[2] and the general solution of (5.8) can only differ from (5.10) by an additive constant (why?). The theorem now follows by combining (5.9) and (5.10).

Remark. Suppose G is multiply connected. Then the integral (5.6) will in general take different values for different paths joining (x_0, y_0) to (x, y), giving rise to a multiple-valued function $v(x, y)$, unless the path is confined to a simply connected subdomain of G. Let D be such a subdomain. Then, just as in the proof of Theorem I.13.7, (5.6) defined single-valued branches of $v(x, y)$ on D, where the particular branch depends on the path taken from (x_0, y_0) to some fixed intermediate point (x_1, y_1) in D, and any two branches differ only by a real constant [cf. (I.13.37)]. In other words, if G is multiply connected, Theorem 5.2 remains valid if we replace G by any simply connected subdomain $D \subset G$ and interpret (5.6) as any single-valued branch of $v(x, y)$ on D. Correspondingly, we interpret (5.7) as any single-valued branch of $f(z)$ on D, where $f(z)$ is now multiple-valued. It should be noted, however, that all the branches of $f(z)$ have the same derivative

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y},$$

i.e., $f'(z)$ is single-valued on G.

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[2] See e.g., D. V. Widder, *op. cit.*, Theorem 4, p. 227 and Theorem 5, p. 229, where it is assumed that L is a piecewise smooth Jordan curve. Actually, L can be an arbitrary rectifiable curve (see Problem 5.4).

23. Poisson's Integral. Schwarz's Formula

We can use the correspondence between harmonic functions and analytic functions, given by Theorems 5.1 and 5.2, to deduce various properties of harmonic functions from familiar properties of analytic functions. Our first result along these lines is the analogue of Cauchy's theorem on the expansion of an analytic function in power series (Theorem I.16.7):

THEOREM 5.3. *Let $u(x, y)$ be a harmonic function on a domain G, with harmonic conjugate $v(x, y)$, let z_0 be an arbitrary (finite) point of G, and let $\Delta = \Delta(z_0)$ be the distance between z_0 and the boundary of G. Then $u(x, y)$ and $v(x, y)$ have expansions of the form*

$$u(x, y) = u(r, \theta) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n, \tag{5.11}$$

$$v(x, y) = v(r, \theta) = \beta_0 + \sum_{n=1}^{\infty} (\beta_n \cos n\theta + \alpha_n \sin n\theta) r^n \tag{5.12}$$

on the disk $|z - z_0| < \Delta$,[3] where $z - z_0 = re^{i\theta}$.

Proof. Using formula (5.7), we form the function $f(z)$ which is analytic on G and has $u(x, y)$ as its real part. According to Theorem I.16.7, $f(z)$ has the power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{5.13}$$

on the disk $|z - z_0| < \Delta$. To obtain (5.11), we substitute

$$a_n = \alpha_n + i\beta_n, \qquad z - z_0 = re^{i\theta}$$

into (5.13) and then take the real part of the resulting equation

$$f(z) = \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) r^n e^{in\theta}. \tag{5.14}$$

To obtain (5.12), we take the imaginary part of (5.14).

Our next result is the analogue of the Cauchy-Hadamard theorem (Theorem I.16.1):

THEOREM 5.4. *Given two series of the form* (5.11) *and* (5.12), *let*

$$R = \frac{1}{\overline{\lim_{n \to \infty}} \sqrt[n]{|\alpha_n + i\beta_n|}}. \tag{5.15}$$

---

[3] If the boundary of G is empty or consists of the single point ∞, we write $\Delta = \infty$. The numbers α_n and β_n are real. Also recall footnote 1, p. 145.

Then (5.11) *and* (5.12) *converge uniformly on every compact subset of the disk* $|z - z_0| < R$ *to a pair of conjugate harmonic functions. The series do not converge on any larger disk.*

Proof. In addition to Theorem I.16.1, use Theorems I.16.3 and I.16.4.[4]

Below, we shall need the following

LEMMA. *The series*

$$\begin{aligned}\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)} &= 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n \cos n(\theta - \varphi),\\ \frac{2\rho r\sin(\theta - \varphi)}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)} &= 2\sum_{n=1}^{\infty}\left(\frac{r}{\rho}\right)^n \sin n(\theta - \varphi)\end{aligned} \tag{5.16}$$

converge uniformly on every compact subset of the disk $|z - z_0| < \rho$.

Proof. Starting from

$$\begin{aligned}\frac{\rho e^{i\varphi} + (z - z_0)}{\rho e^{i\varphi} - (z - z_0)} &= -1 + \frac{2\rho e^{i\varphi}}{\rho e^{i\varphi} - (z - z_0)}\\ &= -1 + 2\left[1 + \frac{z - z_0}{\rho e^{i\varphi}} + \frac{(z - z_0)^2}{\rho^2 e^{2i\varphi}} + \cdots\right]\\ &= 1 + 2\sum_{n=1}^{\infty}\frac{(z - z_0)^n}{\rho^n}e^{-in\varphi},\end{aligned}$$

we set $z - z_0 = re^{i\theta}$ $(r < \rho)$ and use Example 4, p. 146 to take real and imaginary parts of the resulting equation.

The integral

$$\frac{1}{2\pi}\int_0^{2\pi}\mu(\rho, \varphi)\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\,d\varphi,$$

with the "kernel"

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)} = \operatorname{Re}\frac{\rho e^{i\varphi} + (z - z_0)}{\rho e^{i\varphi} - (z - z_0)},$$

is called *Poisson's integral.* As we now show, a characteristic feature of harmonic functions is that they can be represented as Poisson integrals:

THEOREM 5.5. *With the same notation as in Theorem* 5.3,

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi}u(\rho, \varphi)\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\,d\varphi, \tag{5.17}$$

$$v(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi}v(\rho, \varphi)\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\,d\varphi \tag{5.18}$$

[4] Obviously, the number Δ figuring in Theorem 5.3 cannot exceed R.

for $r < \rho < \Delta$ *and arbitrary* θ. *Moreover*

$$v(r, \theta) = \beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \varphi)\,\frac{2\rho r \sin(\theta - \varphi)}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\,d\varphi, \tag{5.19}$$

in terms of $u(r, \theta)$.

Proof. We start from formula (5.11), with r replaced by ρ ($\rho < \Delta$), θ replaced by φ, and n replaced by m:

$$u(\rho, \varphi) = \alpha_0 + \sum_{m=1}^{\infty} (\alpha_m \cos m\varphi - \beta_m \sin m\varphi)\rho^m. \tag{5.20}$$

Using the uniform convergence of (5.20) in φ for every $\rho < \Delta$, we multiply (5.20) by $\cos n\varphi$ and integrate term by term with respect to φ between 0 and 2π. The result is

$$\begin{aligned} \alpha_0 &= \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \varphi)\,d\varphi, \\ \alpha_n &= \frac{1}{\pi\rho^n}\int_0^{2\pi} u(\rho, \varphi)\cos n\varphi\,d\varphi \qquad (n \geqslant 1). \end{aligned} \tag{5.21}$$

Similarly, multiplying (5.20) by $\sin n\varphi$ and integrating term by term with respect to φ between the same limits, we obtain

$$-\beta_n = \frac{1}{\pi\rho^n}\int_0^{2\pi} u(\rho, \varphi)\sin n\varphi\,d\varphi \qquad (n \geqslant 1). \tag{5.22}$$

Substitution of (5.21) and (5.22) into (5.11) and (5.12) gives

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \varphi)\,d\varphi + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_0^{2\pi} u(\rho, \varphi)\left(\frac{r}{\rho}\right)^n \cos n(\theta - \varphi)\,d\varphi, \\ v(r, \theta) &= \beta_0 + \sum_{n=1}^{\infty}\frac{1}{\pi}\int_0^{2\pi} u(\rho, \varphi)\left(\frac{r}{\rho}\right)^n \sin n(\theta - \rho)\,d\varphi. \end{aligned} \tag{5.23}$$

Formulas (5.17) and (5.19) now follow at once from the lemma, after multiplying the series (5.16) by

$$\frac{1}{2\pi}u(\rho, \varphi),$$

integrating term by term with respect to φ from 0 to 2π for fixed r and ρ ($r < \rho$), and comparing the results with (5.23).[5] Moreover, since (5.17) holds for an *arbitrary* harmonic function on G, we can replace $u(r, \theta)$ by $v(r, \theta)$, obtaining (5.18).

Another proof of formulas (5.17) and (5.18) exploits the analogy between Poisson's integral and Cauchy's integral. Let γ_ρ be the circle

[5] In keeping with Theorem 5.2, $v(r, \theta)$ is defined only to within an arbitrary real constant β_0.

of radius ρ with center z_0, and let $z \in I(\gamma_\rho)$. Inverting the point z in the circle γ_ρ, we obtain the point

$$z^* = z_0 + \frac{\rho^2}{\bar{z} - \bar{z}_0}$$

[cf. formula (I.10.32)], where $z^* \in E(\gamma_\rho)$. It follows that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$
$$0 = \frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z^*}\,d\zeta$$

(cf. Theorem I.14.1). Subtracting the second of these equations from the first, we find that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_\rho} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - z^*}\right] d\zeta,$$

which becomes

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left[\frac{\rho}{\rho - re^{i(\theta-\varphi)}} + \frac{re^{i(\varphi-\theta)}}{\rho - re^{i(\varphi-\theta)}}\right] d\varphi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\,\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\,d\varphi,$$

if we write

$$\zeta - z = \zeta - z_0 - (z - z_0) = \rho e^{i\varphi} - re^{i\theta},$$
$$\zeta - z^* = \zeta - z_0 - (z^* - z_0) = \rho e^{i\varphi} - \frac{\rho^2}{r}e^{i\theta}.$$

Finally, replacing $f(z)$ by $u(r, \theta) + iv(r, \theta)$ and $f(\zeta)$ by $u(\rho, \varphi) + iv(\rho, \varphi)$, and taking real and imaginary parts, we obtain (5.17) and (5.18).

COROLLARY. *With the same notation as in Theorem* 5.3, *let* $f(z)$ *be an analytic function on* G *with* $u(r, \theta)$ *as its real part. Then*

$$f(z) = i\beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \varphi)\,\frac{\rho e^{i\varphi} + (z - z_0)}{\rho e^{i\varphi} - (z - z_0)}\,d\varphi \qquad (5.24)$$

in terms of $u(r, \theta)$, *a representation known as Schwarz's formula.*[6]

Proof. Multiply (5.19) by i and add the result to (5.17). Then recall Example 4, p. 146.

Although a trivial consequence of Theorem 5.5, the next result is sufficiently important to warrant being called a theorem:

THEOREM 5.6. *With the same notation as in Theorem* 5.3,

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \varphi)\,d\varphi, \qquad (5.25)$$

[6] Again in keeping with Theorem 5.2, $f(z)$ is defined only to within an arbitrary purely imaginary constant $i\beta_0 = \operatorname{Im} f(z_0)$.

where $z_0 = (x_0, y_0)$, $\rho < \Delta(z_0)$, *or more explicitly,*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + \rho \cos \varphi, y_0 + \rho \sin \varphi)\, d\varphi.$$

In other words, the value of a harmonic function $u(x, y)$ *at the point* (x_0, y_0) *equals the average of its values on the circle* $|z - z_0| = \rho$ *with center* (x_0, y_0).[7]

Proof. Set $r = 0$ in formula (5.17). Alternatively, take the real part of formula (I.14.5).

Remark. Setting $u(r, \theta) \equiv 1$ in (5.17), we obtain the formula

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}\, d\varphi = 1, \tag{5.26}$$

which will be needed later.

Finally, we establish some results on harmonic functions implied by the maximum and minimum modulus principles:

THEOREM 5.7. *Let* $u(x, y) \not\equiv$ const *be harmonic on a domain* G. *Then* $u(x, y)$ *has neither a maximum nor a minimum at any point of* G.

Proof. Let $z_0 = x_0 + iy_0$ be an arbitrary point of G, and consider any neighborhood $\mathcal{N}(z_0) \subset G$. Use Theorem 5.2 to form a function $f(z) = u(x, y) + iv(x, y)$ which is single-valued and analytic on $\mathcal{N}(z_0)$. Then

$$g(z) = e^{f(z)}$$

is analytic on $\mathcal{N}(z_0)$, $g(z) \not\equiv$ const [since $u(x, y) \not\equiv$ const], and

$$|g(z)| = e^{u(x, y)}.$$

The function $u(x, y)$ cannot have a maximum at (x_0, y_0), since otherwise $|g(z)|$ would have a maximum at z_0, thereby contradicting the maximum modulus principle (Theorem I.17.5 or Theorem 3.5, Corollary 2). Similarly, $u(x, y)$ cannot have a minimum at (x_0, y_0), since otherwise $|g(z)|$ would have a minimum but not a zero at z_0, thereby contradicting the minimum modulus principle (Theorem I.17.5, Corollary or Theorem 3.5, Corollary 3).

Alternatively, we can give a proof starting directly from Theorem 3.5. The analytic function $w = u + iv = f(z)$ maps $\mathcal{N}(z_0)$ onto some domain $\mathcal{G}$ in the w-plane, containing the point $w_0 = f(z_0)$. Obviously, $\mathcal{G}$ contains both points with abscissas larger than $u(x_0, y_0)$ and points with abscissas smaller than $u(x_0, y_0)$, i.e., $u(x, y)$ can have neither a

[7] This result is the analogue of Theorem I.14.2. The converse proposition will be proved below (see Theorem 7.1).

maximum nor a minimum at (x_0, y_0). This method of proof has been anticipated in Problem 3.4.

COROLLARY 1. *Let $u(x, y)$ be harmonic on a domain G and continuous on $\bar{G}$. Then $u(x, y)$ achieves its maximum and its minimum on the boundary of G.*

Proof. Since $u(x, y)$ is continuous on the compact set $\bar{G}$, it achieves both its maximum and its minimum on $\bar{G}$ (cf. Theorem I.3.16). But $u(x, y)$ cannot have a maximum or a minimum at a point of G.

COROLLARY 2. *Let $u(x, y)$ be harmonic on a domain G and continuous on $\bar{G}$, and suppose $u(x, y) = \text{const}$ on the boundary of G. Then $u(x, y) = \text{const}$ on $\bar{G}$.*

Proof. Apply Corollary 1.

COROLLARY 3. *Let $u_1(x, y)$ and $u_2(x, y)$ be two functions harmonic on a domain G and continuous on $\bar{G}$, and suppose $u_1(x, y) = u_2(x, y)$ on the boundary of G. Then $u_1(x, y) = u_2(x, y)$ on $\bar{G}$.*

Proof. The function

$$u(x, y) = u_1(x, y) - u_2(x, y)$$

satisfies the conditions of Corollary 2, where the constant equals zero.

24. The Dirichlet Problem for a Disk

Let G be a domain with boundary Γ, and let $\mu(z) = \mu(x, y)$ be a continuous real function defined on Γ. Consider the problem of finding a function $u(x, y)$ such that

1. $u(x, y)$ is harmonic on G and continuous on $\bar{G}$;
2. $u(x, y) = \mu(x, y)$ at every point of Γ.

This is the *Dirichlet problem*, of great importance in mathematical physics as well as in function theory.

Remark. To give a physical interpretation of the Dirichlet problem, imagine a rigid wire frame, in the shape of some closed curve, supporting a stretched membrane whose boundary is fastened to the frame.[8] Suppose the

[8] By a *membrane* we mean a thin sheet whose thickness is negligible compared to its other dimensions, and whose resistance to bending is negligible compared to the tension forces applied to the sheet at its boundary. (These forces are normal to the supporting frame and tangential to the surface of the sheet.) One might think in terms of a thin soap film.

projection of the frame onto the xy-plane is a closed Jordan curve Γ, with interior $I(\Gamma)$ [see Figure 5.1], and suppose the membrane, regarded as a surface, has the equation

$$u = u(x, y)$$

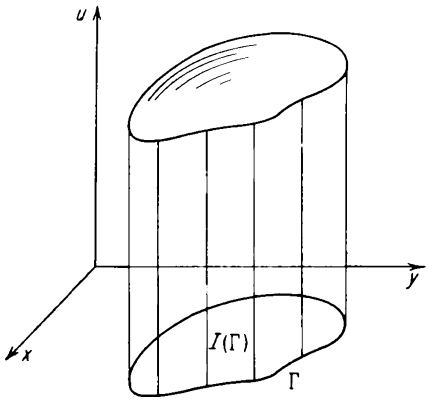

FIGURE 5.1

where $u(x, y)$ is one-to-one on $\overline{I(\Gamma)}$. Clearly, the values of $u(x, y)$ on Γ depend on the position of the frame; let these boundary values be denoted by $\mu(x, y)$. Then, as is well known,[9] the equilibrium position of the membrane in the absence of an external load (and of any effect due to the membrane's own weight) is given by the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which satisfies the boundary condition $u(x, y) = \mu(x, y)$ at every point of Γ. In other words, the problem of determining the equilibrium position of a stretched unloaded membrane reduces to finding a function harmonic on $I(\Gamma)$ and continuous on $\overline{I(\Gamma)}$, which takes given values $\mu(x, y)$ on Γ. But this is just the Dirichlet problem for $G = I(\Gamma)$. Conversely, any given Dirichlet problem can be interpreted in terms of finding the equilibrium position of a stretched membrane, supported by a rigid frame corresponding to the given boundary conditions.

We now consider the Dirichlet problem for the simple but important case where the domain G is a disk $|z - z_0| < \rho$ with circular boundary $\Gamma: |z - z_0| = \rho$.[10] Then the function $\mu(z)$ takes the form

$$\mu(z) = \mu(z_0 + \rho e^{i\varphi}) = \mu(\varphi) \qquad (0 \leqslant \varphi \leqslant 2\pi),$$

where $\mu(\varphi)$ must obviously satisfy the condition $\mu(0) = \mu(2\pi)$. If $\mu(\varphi)$ coincides with the values $u(\rho, \varphi)$ taken on Γ by some function $u(r, \theta)$ which is harmonic on a disk $|z - z_0| < \rho'$ of larger radius than G (so that $\rho' > \rho$), then, since

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho, \varphi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} \, d\varphi$$

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[9] See e.g., R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volume I*, Interscience Publishers, New York (1953), p. 247.

[10] The Dirichlet problem for more general domains will be considered at the end of Sec. 27.

(see Theorem 5.5), the Dirichlet problem has the obvious solution

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \mu(\varphi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} d\varphi \tag{5.27}$$

in terms of Poisson's integral, and moreover this solution is unique, by Theorem 5.7, Corollary 3. However, the true nature of the problem is apparent only when the function $\mu(\varphi)$ is specified *a priori*, without reference to a function already known to be harmonic on a larger disk than G. As might be suspected, even in this case, the solution of the Dirichlet problem is still given by (5.27). In fact, we have the following

THEOREM 5.8 (*Dirichlet problem for a disk*). *Let G be the disk $|z - z_0| < \rho$ with boundary $\Gamma: |z - z_0| = \rho$, and let $\mu(\varphi)$ be a continuous real function on the interval $[0, 2\pi]$ such that $\mu(0) = \mu(2\pi)$. Then the function $u(r, \theta)$ defined by the integral* (5.27) *for any point (r, θ) in G, and by*

$$u(\rho, \varphi) = \mu(\varphi)$$

for any point (ρ, φ) in Γ, solves the Dirichlet problem for the domain G.

Proof. First we prove that the integral (5.27) defines a harmonic function on G. The reasoning here is essentially the reverse of that used to prove Theorem 5.5. According to the lemma on p. 149, if $r < \rho$ we have[11]

$$\begin{aligned} u(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \mu(\varphi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mu(\varphi)\, d\varphi + \sum_{n=1}^{\infty} \frac{1}{\pi} \int_0^{2\pi} \mu(\varphi) \left(\frac{r}{\rho}\right)^n \cos n(\theta - \varphi)\, d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mu(\varphi)\, d\varphi + \sum_{n=1}^{\infty} \left\{ \left[\frac{1}{\pi \rho^n} \int_0^{2\pi} \mu(\varphi) \cos n\varphi\, d\varphi \right] r^n \cos n\theta \right. \\ &\qquad \left. + \left[\frac{1}{\pi \rho^n} \int_0^{2\pi} \mu(\varphi) \sin n\varphi\, d\varphi \right] r^n \sin n\theta \right\} \\ &= \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n \end{aligned} \tag{5.28}$$

where

$$\alpha_0 = \frac{1}{2\pi} \int_0^{2\pi} \mu(\varphi)\, d\varphi, \quad \alpha_n = \frac{1}{\pi \rho^n} \int_0^{2\pi} \mu(\varphi) \cos n\varphi\, d\varphi \qquad (n \geqslant 1), \tag{5.29}$$

$$-\beta_n = \frac{1}{\pi \rho^n} \int_0^{2\pi} \mu(\varphi) \sin n\varphi\, d\varphi \qquad (n \geqslant 1)$$

[11] Multiply the first of the series (5.16) by $(1/2\pi)\mu(\varphi)$, and integrate term by term with respect to φ from 0 to 2π.

[cf. (5.21) and (5.22)]. In particular,

$$|\alpha_n + i\beta_n| = \left|\frac{1}{\pi\rho^n}\int_0^{2\pi} \mu(\varphi)e^{-in\varphi}\,d\varphi\right| \leqslant \frac{1}{\pi\rho^n}\int_0^{2\pi} |\mu(\varphi)|\,d\varphi,$$

and therefore

$$\rho \leqslant \lim_{n\to\infty} \frac{\sqrt[n]{\frac{1}{\pi}\int_0^{2\pi} |\mu(\varphi)|\,d\varphi}}{\sqrt[n]{|\alpha_n + i\beta_n|}} = \frac{1}{\overline{\lim_{n\to\infty}}\sqrt[n]{|\alpha_n + i\beta_n|}} = R,$$

where R is the number given by (5.15). It follows from the representation (5.28) and Theorem 5.4 that $u(r, \theta)$ is harmonic on G.

Next we prove that the function $u(r, \theta)$ defined by the integral (5.27) approaches $\mu(\varphi_0)$ as the point (r, θ) in G approaches any fixed point (ρ, φ_0) in Γ. Using (5.26), we form the difference

$$u(r_n, \theta_n) - \mu(\varphi_0) = \frac{1}{2\pi}\int_0^{2\pi} [\mu(\varphi) - \mu(\varphi_0)]\frac{\rho^2 - r_n^2}{\rho^2 + r_n^2 - 2\rho r_n\cos(\theta_n - \varphi)}\,d\varphi, \tag{5.30}$$

where (r_n, θ_n) is an arbitrary sequence of points in G converging to the boundary point (ρ, φ_0). Since $\mu(\varphi)$ is (uniformly) continuous on Γ, given any $\varepsilon > 0$, there is a number $\delta = \delta(\varepsilon) > 0$ such that

$$|\mu(\varphi) - \mu(\varphi_0)| < \frac{\varepsilon}{2} \tag{5.31}$$

whenever

$$|\varphi - \varphi_0| < 2\delta. \tag{5.32}$$

Assuming that δ satisfies (5.32) and is also so small that $\varphi_0 - 2\delta > 0$ and $\varphi_0 + 2\delta < 2\pi$,[12] we write

$$\begin{aligned} |u(r_n, \theta_n) - \mu(\varphi_0)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \cdots\right| \\ &\leqslant \left|\frac{1}{2\pi}\int_0^{\varphi_0 - 2\delta} \cdots\right| + \left|\frac{1}{2\pi}\int_{\varphi_0 - 2\delta}^{\varphi_0 + 2\delta} \cdots\right| + \left|\frac{1}{2\pi}\int_{\varphi_0 + 2\delta}^{2\pi} \cdots\right|, \end{aligned} \tag{5.33}$$

where in each case the dots denote the rest of the integral (5.30). For the middle term in the right-hand side of (5.33), we have the estimate

$$\left|\frac{1}{2\pi}\int_{\varphi_0 - 2\delta}^{\varphi_0 + 2\delta} \cdots\right| < \frac{\varepsilon}{2}\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r_n^2}{\rho^2 + r_n^2 - 2\rho r_n\cos(\theta_n - \varphi)}\,d\varphi = \frac{\varepsilon}{2}, \tag{5.34}$$

[12] There is no need to consider the case $\varphi_0 = 0$ separately, since the axis of our polar coordinate system (from which the angles θ and φ are measured) can always be rotated to make $\varphi_0 > 0$.

where we have used (5.31) and (5.26). To estimate the other two terms, we choose n so large that $|\theta_n - \varphi_0| < \delta$. Then

$$\left|\frac{1}{2\pi}\int_0^{\varphi_0-2\delta}\cdots\right| + \left|\frac{1}{2\pi}\int_{\varphi_0+2\delta}^{2\pi}\cdots\right|$$
$$\leqslant 2M\,\frac{1}{2\pi}\,\frac{\rho^2 - r_n^2}{\rho^2 + r_n^2 - 2\rho r_n\cos\delta}\left(\int_0^{\varphi_0-2\delta} d\varphi + \int_{\varphi_0+2\delta}^{2\pi} d\varphi\right)$$
$$< 2M\,\frac{\rho^2 - r_n^2}{\rho^2 + r_n^2 - 2\rho r_n\cos\delta}, \tag{5.35}$$

where

$$M = \max_{\varphi\in[0,\,2\pi]} |\mu(\varphi)|,$$

and we have used the fact that

$$\cos(\theta_n - \varphi) < \cos\delta,$$
$$\frac{\rho^2 - r_n^2}{\rho^2 + r_n^2 - 2\rho r_n\cos(\theta_n - \varphi)} < \frac{\rho^2 - r_n^2}{\rho^2 + r_n^2 - 2\,\rho r\cos\delta}$$

if $|\theta_n - \varphi_0| < \delta$ and φ belongs to either of the intervals $[0, \varphi_0 - 2\delta]$ or $[\varphi_0 + 2\delta, 2\pi]$. The right-hand side of (5.35) obviously approaches zero as $r_n \to \rho$. Therefore, for sufficiently large n,

$$\left|\frac{1}{2\pi}\int_0^{\varphi_0-2\delta}\cdots\right| + \left|\frac{1}{2\pi}\int_{\varphi_0+2\delta}^{2\pi}\cdots\right| < \frac{\varepsilon}{2} \tag{5.36}$$

(naturally, we assume that n is also so large that $|\theta_n - \varphi_0| < \delta$). Comparing (5.33), (5.35) and (5.36), we find that

$$|u(r_n, \theta_n) - \mu(\varphi_0)| < \varepsilon$$

for sufficiently large n, i.e.,

$$\lim_{n\to\infty}[u(r_n, \theta_n) - \mu(\varphi_0)] = 0, \tag{5.37}$$

where (r_n, θ_n) is an arbitrary sequence of points in G approaching (ρ, φ_0). Equation (5.37) still holds if some or all of the points (r_n, θ_n) lie on Γ, since then we can make direct use of the continuity of $\mu(\varphi)$ [recall that $u(\rho, \varphi) = \mu(\varphi)$, by definition]. In other words, $u(r, \theta)$ is continuous on $\bar{G}$, and the proof is complete.

COROLLARY 1. *Let $\mu(\varphi)$ be a continuous real function in the interval $[0, 2\pi]$ such that $\mu(0) = \mu(2\pi)$, and let*

$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\,d\varphi, \quad \alpha_n = \frac{1}{\pi}\int_0^{2\pi}\mu(\varphi)\cos n\varphi\,d\varphi \qquad (n \geqslant 1),$$
$$\beta_n = \frac{1}{\pi}\int_0^{2\pi}\mu(\varphi)\sin n\varphi\,d\varphi \qquad (n \geqslant 1). \tag{5.38}$$

Then the formal Fourier series

$$\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n\cos n\varphi + \beta_n\sin n\varphi)$$

is uniformly summable by Abel's method to $\mu(\varphi)$,[13] *i.e., as* $r \to 1-$, *the function*

$$\mu(r, \varphi) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\varphi + \beta_n \sin n\varphi) r^n \qquad (0 \leqslant r < 1) \quad (5.39)$$

converges uniformly to $\mu(\varphi)$ *in* $[0, 2\pi]$.

Proof. First we note that the *series* (5.39) is uniformly convergent on every compact subset of the disk $r < 1$ (why?). The result of setting $\rho = 1$ in (5.29) is identical with (5.38), except for a minus sign. We now form the analogue of (5.28), with $0 = \varphi$, and let $r \to \rho = 1$ from the left.

COROLLARY 2 (*Weierstrass' approximation theorem*). *Let* $\mu(\varphi)$ *be a continuous real function in the interval* $[0, 2\pi]$ *such that* $\mu(0) = \mu(2\pi)$. *Then, given any* $\varepsilon > 0$, *there is a trigonometric polynomial*

$$\tau(\varphi) = \sum_{n=0}^{N} (a_n \cos n\varphi + b_n \sin n\varphi)$$

such that

$$|\tau(\varphi) - \mu(\varphi)| < \varepsilon \qquad (0 \leqslant \varphi \leqslant 2\pi).$$

Proof. Let $\mu_N(r, \varphi)$ denote the Nth partial sum of the series (5.39). According to Corollary 1,

$$|\mu(r_0, \varphi) - \mu(\varphi)| < \frac{\varepsilon}{2} \qquad (0 \leqslant \varphi \leqslant 2\pi)$$

for some r_0 $(0 \leqslant r_0 < 1)$. Moreover, by the uniform convergence of (5.39),

$$|\mu_N(r_0, \varphi) - \mu(r_0, \varphi)| < \frac{\varepsilon}{2}$$

for some integer N, and hence

$$|\mu_N(r_0, \varphi) - \mu(\varphi)| < \varepsilon \qquad (0 \leqslant \varphi \leqslant 2\pi).$$

Now choose

$$\tau(\varphi) = \mu_N(r_0, \varphi).$$

25. Behavior of a Harmonic Function near an Isolated Singular Point

A single-valued harmonic function which is harmonic in a deleted neighborhood $\mathcal{N}'(z_0)$, but not in any neighborhood $\mathcal{N}(z_0)$ containing z_0 itself, is said to have an *isolated singular point* at z_0. Without loss of generality,

[13] See e.g., G. P. Tolstov, *Fourier Series* (translated by R. A. Silverman), Prentice-Hall, Inc., Englewood Cliffs, N. J. (1962), Chap. 6, in particular, Theorem 3, p. 169. A Fourier series may be summable by Abel's method even at points where it diverges.

we can assume that $z_0 = 0$ (this can always be achieved by making the preliminary transformation $z^* = z - z_0$). Thus, starting from a single-valued function $u(x, y) = u(r, \theta)$ which is harmonic in a deleted neighborhood $\mathcal{N}'(0)$, we construct the conjugate harmonic function $v(x, y)$ and the analytic function $f(z) = u(x, y) + iv(x, y)$ given by formulas (5.6) and (5.7). In general, $v(x, y)$ and $f(z)$ will be multiple-valued, in the way described in the remark on p. 147, but in any event, the derivative

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

will be a single-valued analytic function, with a Laurent expansion

$$f'(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

and hence

$$f(z) = C + a_{-1} \operatorname{Ln} z + \sum_{n=-\infty}^{\infty}{}' \frac{a_n z^{n+1}}{n+1},$$

where the prime on the summation sign indicates that the value $n = -1$ is omitted from the sum. It follows that

$$u(r, \theta) = \operatorname{Re} f(z) = \operatorname{Re} C + \operatorname{Re} a_{-1} \cdot \ln r - \operatorname{Im} a_{-1} \cdot \theta + \sum_{n=-\infty}^{\infty}{}' \left[\frac{\operatorname{Re} a_n}{n+1} \cos (n+1)\theta - \frac{\operatorname{Im} a_n}{n+1} \sin (n+1)\theta\right] r^{n+1}, \tag{5.40}$$

where $\theta = \operatorname{Arg} z$. Since $u(r, \theta)$ is single-valued, we must have $\operatorname{Im} a_{-1} = 0$, and then (5.40) takes the form

$$u(r, \theta) = c \ln r + \sum_{n=-\infty}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta) r^n \tag{5.41}$$

(the changes in notation are self-explanatory), i.e., $u(r, \theta)$ is the real part of the analytic function

$$f(z) = c \operatorname{Ln} z + \sum_{n=-\infty}^{\infty} (\alpha_n - i\beta_n) z^n = c \operatorname{Ln} z + \psi(z),$$

where c is real.

If the point $z = 0$ is an isolated singular point of $u(r, \theta)$, then it must also be an isolated singular point of $f(z)$ [why?]. *A priori*, the point $z = 0$ may either be an isolated singular point of $\psi(z)$ [i.e., a pole or an essential singular point] or a regular point of $\psi(z)$. In the second case, c is necessarily nonzero, since otherwise $z = 0$ would be a regular point of $f(z)$. We now examine these two cases in turn:

Case 1. If $z = 0$ is a singular point of $\psi(z)$, at least one of the coefficients $\alpha_{-1}, \beta_{-1}, \ldots, \alpha_{-n}, \beta_{-n}, \ldots$ is nonzero. To be explicit, suppose $\alpha_{-m} \neq 0$ $(m \geqslant 1)$. Then, as we now show, $u(x, y)$ takes all real values in every deleted

neighborhood of the origin. In fact, since $u(x, y)$ is continuous for $0 < |z| < \varepsilon$, and hence cannot go from one value to another without taking all intermediate values (see Problem I.4.11), it is enough to prove that $u(x, y)$ takes positive and negative values of arbitrarily large absolute value. Suppose the contrary, so that $u(x, y)$ does not take arbitrarily large positive values (say), i.e., $u(x, y)$ is bounded from above:

$$u(x, y) \leqslant M.$$

[If $u(x, y)$ is bounded from below, consider the function $-u(x, y)$ instead.] It follows from (5.41) that

$$\int_0^{2\pi} u(r, \theta) \cos m\theta \, d\theta = \pi(\alpha_m r^m + \alpha_{-m} r^{-m}),$$

while

$$\int_0^{2\pi} u(r, \theta) \, d\theta = 2\pi(\alpha_0 + c \ln r).$$

Therefore

$$\pi[2(\alpha_0 + c \ln r) \pm (\alpha_m r^m + \alpha_{-m} r^{-m})] = \int_0^{2\pi} u(r, \theta)(1 \pm \cos m\theta) \, d\theta \leqslant 4\pi M,$$

and choosing the sign $\pm$ so that $\pm\alpha_{-m} = |\alpha_{-m}|$, we have

$$(|\alpha_{-m}| r^{-m} + 2c \ln r) + 2\alpha_0 \pm \alpha_m r^m \leqslant 4M$$

for all r $(0 < r < \varepsilon)$. But this is impossible, since $|\alpha_{-m}| r^{-m} + 2c \ln r \to +\infty$ as $r \to 0$, and the desired result is proved by contradiction (there is an analogous argument for the case where $\beta_{-m} \neq 0$).

Case 2. If $z = 0$ is a regular point of $\psi(z)$, the expansion (5.41) takes the form

$$u(r, \theta) = c \ln r + \sum_{n=0}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta) r^n, \tag{5.42}$$

where the difference $u(r, \theta) - c \ln r$ is a function harmonic in a neighborhood of $z = 0$. Therefore, as $r \to 0$, $u(r, \theta) \to +\infty$ if $c < 0$ and $u(r, \theta) \to -\infty$ if $c > 0$. In this case, the point $z = 0$ is called a *logarithmic pole* of the given harmonic function $u(r, \theta)$, and the term $c \ln r$ is called the *principal part* of $u(r, \theta)$ at the logarithmic pole.

The following theorem summarizes the result just proved:

THEOREM 5.9. *Let $u(x, y) = u(r, \theta)$ be a single-valued harmonic function with an isolated singular point at $z = 0$. Then there are only two possibilities:*

1. *$u(r, \theta)$ has an expansion of the form* (5.41), *with at least one nonzero coefficient $\alpha_{-1}, \beta_{-1}, \ldots, \alpha_{-n}, \beta_{-n}, \ldots$, and takes all real values in every deleted neighborhood of $z = 0$;*

2. *$u(r, \theta)$ has an expansion of the form* (5.42), *with $c \neq 0$, and approaches $\pm\infty$ as $z = re^{i\theta} \to 0$, where the limit has the sign opposite to c.*

COROLLARY. *If a single-valued function* $u(x, y) = u(r, \theta)$ *is harmonic and bounded in a deleted neighborhood of the point* $z = 0$, *then* $u(r, \theta)$ *has an expansion of the form*

$$u(r, \theta) = \sum_{n=0}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta),$$

i.e., $z = 0$ *is not a singular point of* $u(r, \theta)$.[14]

*26. Sequences of Harmonic Functions. Harnack's Theorem

This section is devoted to some general results involving sequences of harmonic functions:

THEOREM 5.10. *Given a domain* G *and a sequence* $\{f_n(z)\}$ *of analytic functions on* G, *suppose the sequence*

$$\{u_n(x, y)\} = \{\mathrm{Re}\, f_n(z)\}$$

converges uniformly on every compact subset of G, *and suppose* $\{f_n(z)\}$ *converges at some point* $z_0 = (x_0, y_0) \in G$. *Then* $\{f_n(z)\}$ *also converges uniformly on every compact subset of* G.

Proof. Let $\Delta(z)$ be the distance between an arbitrary point $z \in G$ and the boundary of G. First we prove that $\{f_n(z)\}$ is uniformly convergent on every compact subset of the disk $|z - z_0| < \Delta(z_0)$. According to Schwarz's formula (5.24),

$$f_n(z) = i\beta_0^{(n)} + \frac{1}{2\pi} \int_0^{2\pi} u_n(\rho, \varphi) \frac{\rho e^{i\varphi} + (z - z_0)}{\rho e^{i\varphi} - (z - z_0)} d\varphi.$$

Choosing values of r and ρ such that $0 < r < \rho < \Delta(z_0)$, we find that

$$\begin{aligned} |f_{n+p}(z) - f_n(z)| &= \left| i(\beta_0^{(n+p)} - \beta_0^{(n)}) \right. \\ &\quad \left. + \frac{1}{2\pi} \int_0^{2\pi} [u_{n+p}(\rho, \varphi) - u_n(\rho, \varphi)] \frac{\rho e^{i\varphi} + (z - z_0)}{\rho e^{i\varphi} - (z - z_0)} d\varphi \right| \\ &\leqslant |\beta_0^{(n+p)} - \beta_0^{(n)}| \\ &\quad + \max_{0 \leqslant \varphi \leqslant 2\pi} |u_{n+p}(\rho, \varphi) - u_n(\rho, \varphi)| \frac{\rho + r}{\rho - r} \end{aligned}$$

for all z in the closed disk $|z - z_0| \leqslant r$, where n and p are arbitrary positive integers. But $\beta_0^{(n)} = \mathrm{Im}\, f_n(z_0)$, where $\{f_n(z_0)\}$ is convergent, and moreover, $\{u_n(x, y)\}$ is uniformly convergent on the circle $|z - z_0| = \rho$. Thus there is an integer $N(\varepsilon) > 0$ such that

$$|f_{n+p}(z) - f_n(z)| < \varepsilon$$

[14] Alternatively, $z = 0$ is a *removable singular point* of $u(r, \theta)$ in the sense of Remark 1, p. 12 [set $u(0, \theta) = 0$].

for all $n > N(\varepsilon)$, $p > 0$ and $|z - z_0| \leqslant r$. It follows that $\{f_n(z)\}$ is uniformly convergent on every compact subset of $|z - z_0| < \Delta(z_0)$.

Now let E be the set of all points in G at which the sequence $\{f_n(z)\}$ converges. Obviously E is nonempty, since $z_0 \in E$. Moreover, by the argument just given, if z belongs to E, so does every point in G whose distance from z is less than $\Delta(z)$. Therefore, according to Theorem I.4.16, E coincides with G. In other words, $\{f_n(z)\}$ converges at every point of G, and hence, using the same argument again, we see that $\{f_n(z)\}$ is uniformly convergent in some neighborhood of every point $z \in G$. It now follows from Theorem I.15.5 (more exactly, its analogue for sequences) that $\{f_n(z)\}$ is uniformly convergent on every compact subset of G, as asserted.

THEOREM 5.11. *If the sequence $\{u_n(x, y)\}$ is uniformly convergent on every compact subset of a domain G, and if every term $u_n(x, y)$ is harmonic on G, then the limit function*

$$u(x, y) = \lim_{n \to \infty} u_n(x, y)$$

is also harmonic on G.

Proof. Clearly, it is enough to show that every point $z_0 = (x_0, y_0) \in G$ has a neighborhood in which the function $u(x, y)$ is harmonic. Let $\Delta(z_0)$ be the same as in the proof of Theorem 5.10, and use formula (5.7) to construct a sequence $\{f_n(z)\}$ of functions analytic on the disk $K: |z - z_0| < \Delta(z_0)$ such that $u_n(x, y) = \operatorname{Re} f_n(z)$. Since $v_n(x_0, y_0) = 0$ for all n by construction [cf. (5.6)], the sequence $\{f_n(z_0)\} = \{u_n(x_0, y_0)\}$ converges. It follows from Theorem 5.10 that $\{f_n(z)\}$ is uniformly convergent on every compact subset of K, and hence, by Weierstrass' theorem on uniformly convergent sequences of analytic functions (Theorem I.15.8),

$$f(z) = \lim_{n \to \infty} f_n(z)$$

is analytic on K. But then $u(x, y) = \operatorname{Re} f(z)$ is harmonic on K, and the proof is complete.[15]

THEOREM 5.12 (*Harnack's theorem*). *Given a domain G and a series*

$$\sum_{n=1}^{\infty} u_n(x, y) \tag{5.43}$$

whose terms are nonnegative harmonic functions on G, suppose the series converges at some point $(x_0, y_0) = z_0 \in G$. Then the series converges uniformly on every compact subset of G.

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[15] Note the analogy between Theorem 5.11 and the first assertion of Theorem I.15.8. For the rest of the analogy, see Problems 5.21 and 5.22.

Proof. If we succeed in showing that (5.43) converges uniformly on every compact subset of the disk $|z - z_0| < \Delta(z_0)$, where $\Delta(z_0)$ has the usual meaning, then (5.43) converges uniformly on every compact subset of G, by exactly the same argument (based on Theorem I.4.16) as in the proof of Theorem 5.10. Choosing values of r and ρ such that $0 < r < \rho < \Delta(z_0)$, and using formula (5.17), we find that

$$u_n(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u_n(\rho, \varphi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} d\varphi.$$

It follows that

$$u_n(r, \theta) \leqslant \max_{0 \leqslant \varphi \leqslant 2\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} \cdot \frac{1}{2\pi} \int_0^{2\pi} u_n(\rho, \varphi)\, d\varphi, \quad (5.44)$$

since $u_n(\rho, \varphi)$ and the kernel of Poisson's integral are both nonnegative. But obviously, for fixed r and ρ ($r < \rho$), the function

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}$$

achieves its maximum when $\theta - \varphi = 2k\pi$, and this maximum equals

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r} = \frac{\rho + r}{\rho - r}.$$

Moreover, according to formula (5.25),

$$\frac{1}{2\pi} \int_0^{2\pi} u_n(\rho, \varphi)\, d\varphi = u_n(x_0, y_0).$$

Therefore (5.44) implies the inequality

$$u_n(r, \theta) \leqslant \frac{\rho + r}{\rho - r} u_n(x_0, y_0) \qquad (n = 1, 2, \ldots)$$

for all z in the closed disk $|z - z_0| \leqslant r$. On the right we have the general term of a convergent series whose terms are nonnegative constants, since, by hypothesis, the series

$$\sum_{n=1}^{\infty} u_n(x_0, y_0)$$

converges. Therefore, by Weierstrass' M-test (Theorem I.15.2), the series (5.43) is uniformly convergent on the disk $|z - z_0| \leqslant r$, and hence on every compact subset of the disk $|z - z_0| < \Delta(z_0)$. The proof is now completed in the way mentioned.

COROLLARY. *Given a domain G and a sequence $\{U_n(x, y)\}$ whose terms are harmonic functions on G such that $U_n(x, y) \leqslant U_{n+1}(x, y)$ for all $(x, y) \in G$, suppose that the sequence converges at some point*

$(x_0, y_0) \in G$. *Then the sequence converges uniformly on every compact subset of* G.

Proof. Use Harnack's theorem, after writing

$$u_1(x, y) = U_1(x, y),$$
$$u_n(x, y) = U_n(x, y) - U_{n-1}(x, y) \qquad (n \geqslant 1).$$

*27. Generalization of Poisson's Integral. The Dirichlet Problem for a Jordan Domain

We now generalize Poisson's integral to the case where the integral is over a curve more general than a circle and the function $u(r, \theta)$ is harmonic on a domain more general than a disk, borrowing the necessary tools from the theory of conformal mapping. Thus we begin by citing without proof three important facts about conformal mapping which will be needed below:[16]

1. *Riemann's mapping theorem.* Given any simply connected domain G in the extended plane, whose boundary contains more than one point, there exists a univalent function $w = f(z)$ which maps G conformally onto the disk $|w| < 1$.[17]

2. *Uniqueness theorem for conformal mapping.* Let G be the same as in Riemann's mapping theorem and let z_0 be an arbitrary point of G. Then there exists a unique function $w = f_{z_0}(z)$ which maps G conformally onto the disk $|w| < 1$ and satisfies the conditions

$$f_{z_0}(z_0) = 0, \qquad f'_{z_0}(z_0) > 0.$$

3. *Boundary correspondence theorem.* Let G be a bounded simply connected domain with boundary Γ, and let $w = f(z)$ be any function mapping G conformally onto the disk $|w| < 1$. Then $w = f(z)$ establishes a homeomorphism between $\overline{G}$ and the closed disk $|w| \leqslant 1$, and hence between Γ and the circle $|w| = 1$, if and only if Γ is a closed Jordan curve (in this case G is called a *Jordan domain*).[18]

We are now in a position to derive the desired generalization of Poisson's

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[16] For the proofs, see Theorems 1.2, 1.3 and 2.20 of Volume III of this course. The uniqueness theorem, unlike Riemann's mapping theorem and the boundary correspondence theorem, is quite easy to prove (see Problem 5.22).

[17] In this context, it is customary to drop the adjective "univalent," i.e., a conformal mapping of one domain onto another is tacitly assumed to be accomplished by a univalent function.

[18] For the definition of a homeomorphism, see p. 1.94. The fact that $w = f(z)$ is homeomorphic on G follows from Theorem 1.6.1. At points of Γ, $f(z)$ is defined by continuity, i.e., if $\zeta \in \Gamma$, then

$$f(\zeta) = \lim_{\substack{z \to \zeta \\ z \in G}} f(z)$$

(it is shown in Volume III that this limit is unique if Γ is a closed Jordan curve).

integral. First we prove the following

LEMMA. *If $u(z) = u(x, y)$ is harmonic on a domain G in the z-plane, and if $w = f(z)$ maps G conformally onto a domain D in the w-plane, then*

$$u^*(w) = u[f^{-1}(w)]$$

is harmonic on D.

Proof. According to Theorem 5.2,

$$u(z) = \operatorname{Re} g(z),$$

where $g(z)$ is analytic on G. But then

$$u^*(w) = u[f^{-1}(w)] = \operatorname{Re} g[f^{-1}(w)],$$

where $g[f^{-1}(w)]$ is analytic on D (why?), i.e., $u^*(w)$ is harmonic on D (see Theorem 5.1).

THEOREM 5.13. *Given a simply connected domain G whose boundary contains more than one point, let $u(z)$ be a function harmonic on G. Moreover, let z_0 be an arbitrary point of G, and let $w = f(z) = f_{z_0}(z)$ be the (unique) function which maps G conformally onto the disk $|w| < 1$ and satisfies the conditions*

$$f(z_0) = 0, \quad f'(z_0) > 0. \tag{5.45}$$

Then the analogue of formula (5.17) is

$$u(z) = \frac{1}{2\pi} \int_{\gamma_\rho} u(\zeta) \frac{|f(\zeta)|^2 - |f(z)|^2}{|f(\zeta) - f(z)|^2} \frac{|f'(\zeta)|}{|f(\zeta)|} d\sigma, \tag{5.46}$$

where γ_ρ is the inverse image of the circle $|w| = \rho < 1$ under the mapping $w = f(z)$, z is an arbitrary point inside γ_ρ, and $d\sigma = |d\zeta|$ is the element of arc length along γ_ρ. In particular,

$$u(z_0) = \frac{1}{2\pi} \int_{\gamma_\rho} u(\zeta) \frac{\partial}{\partial \nu} \ln \frac{1}{|f_{z_0}(\zeta)|} d\sigma \tag{5.47}$$

where $\partial/\partial\nu$ denotes differentiation along the interior normal to γ_ρ.

Proof. According to the lemma,

$$u^*(w) = u[f^{-1}(w)]$$

is harmonic on the disk $|w| < 1$. Therefore (5.17) implies that

$$u(z) = u^*(w) = u^*(r, 0) = \frac{1}{2\pi} \int_0^{2\pi} u^*(\rho, \varphi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(0 - \varphi)} d\varphi. \tag{5.48}$$

But

$$d\varphi = \frac{ds}{\rho} = \frac{|f'(\zeta)|}{|f(\zeta)|} d\sigma, \tag{5.49}$$

where ds is the element of arc length along the circle $|w| = \rho$, while

$d\sigma = |d\zeta|$ is the element of arc length along the curve γ_ρ. Moreover, since

$$f(z) = re^{i\theta}, \qquad f(\zeta) = \rho e^{i\varphi},$$

we have

$$\begin{gathered} |f(z)| = r, \qquad |f(\zeta)| = \rho, \\ f(z)\overline{f(\zeta)} + \overline{f(z)}f(\zeta) = 2\rho r \cos(\theta - \varphi). \end{gathered} \tag{5.50}$$

Using (5.49) and (5.50) to express (5.48) as an integral along γ_ρ, we obtain formula (5.45), which reduces to

$$u(z_0) = \frac{1}{2\pi} \int_{\gamma_\rho} u(\zeta) \frac{|f'(\zeta)|}{|f(\zeta)|} d\sigma \tag{5.51}$$

for $z = z_0$ [cf. (5.45)].[19]

To transform (5.51) into (5.47), let $d\nu$ be the element of arc length along the curve which passes through $\zeta \in \gamma_\rho$ and is the inverse image under $w = f(z)$ of the segment joining the point $f(z_0) = 0$ to the point $f(\zeta)$. Because of the conformality, $d\nu$ is directed along the normal to γ_ρ at ζ; choose this to be the *interior* normal. Then the mapping $w = f(z)$ carries $|f'(\zeta)|\, d\nu$ into $-d\rho$ (where $d\rho < 0$) and $|f(\zeta)|$ into ρ. It follows that

$$\frac{|f'(\zeta)|}{|f(\zeta)|} d\nu = -\frac{d\rho}{\rho} = d \ln \frac{1}{\rho} = d \ln \frac{1}{|f(\zeta)|},$$

and hence

$$\frac{|f'(\zeta)|}{|f(\zeta)|} = \frac{\partial}{\partial \nu} \ln \frac{1}{|f(\zeta)|}. \tag{5.52}$$

Substituting (5.52) into (5.51) and restoring the subscript z_0, we obtain (5.47).

The function

$$g(z, z_0) = \ln \frac{1}{|f_{z_0}(z)|} \tag{5.53}$$

appearing in (5.47) merits further study. Since $f_{z_0}(z)$ has a unique simple zero at $z = z_0$, we can write

$$f_{z_0}(z) = (z - z_0)\psi_{z_0}(z),$$

where $\psi_{z_0}(z)$ is analytic and nonzero on the domain G, and hence

$$\ln \frac{1}{|f_{z_0}(z)|} = \ln \frac{1}{|z - z_0|} + \ln \frac{1}{|\psi_{z_0}(z)|}.$$

Therefore the function $g(z, z_0)$ has the following properties:

1. It is harmonic on G except at $z = z_0$, and has a logarithmic pole at z_0, with principal part $-\ln |z - z_0|$;

[19] We can also write (5.51) directly from (5.25), using (5.49).

2. It is positive on G (since $|f_{z_0}(z)| < 1$ on G) and approaches zero as z approaches the boundary of G (since then the point $w = f(z)$ approaches the unit circle $|w| = 1$).

Such a function $g(z, z_0)$ is called a *Green's function* for the domain G (more precisely, for the Dirichlet problem on G). It is easy to see that $g(z, z_0)$ is uniquely determined. In fact, if $\tilde{g}(z, z_0)$ has the same properties as $g(z, z_0)$, then the function

$$\tilde{g}(z, z_0) - g(z, z_0),$$

being harmonic on G (including the point z_0) and zero on the boundary of G, must vanish identically, by Theorem 5.7, Corollary 2. In terms of the Green's function, (5.47) takes the form

$$u(z_0) = \frac{1}{2\pi} \int_{\gamma_\rho} u(\zeta) \frac{\partial g(\zeta, z_0)}{\partial \nu} \, d\sigma. \tag{5.54}$$

THEOREM 5.14. *The Green's function* $g(z, z_0)$ *is symmetric, i.e.,* $g(z_1, z_0) = g(z_0, z_1)$.

Proof. Choosing another point $z_1 \in G$ we have

$$g(z, z_1) = \ln \frac{1}{|f_{z_1}(z)|}, \tag{5.55}$$

where $f_{z_1}(z)$ is the function which maps G conformally onto the unit disk $|w| < 1$ and satisfies the conditions

$$f_{z_1}(z_1) = 0, \quad f'_{z_1}(z_1) > 0.$$

It is clear that

$$f_{z_1}(z) = e^{i\alpha} \frac{f_{z_0}(z) - f_{z_0}(z_1)}{1 - \overline{f_{z_0}(z_1)} f_{z_0}(z)}, \tag{5.56}$$

where α is chosen so that $f'_{z_1}(z_0) > 0$. To see this, we regard $f_{z_1}(z)$ as the result of the mapping $f_{z_0}(z)$ which carries G into the unit disk, followed by the mapping

$$\eta = e^{i\alpha} \frac{w - w_0}{1 - \overline{w}_0 w}$$

which carries the unit disk into itself and the point w_0 into the point $\eta = 0$ [cf. (I.10.35)]. The appropriate choice of α is $-\operatorname{Arg} f'_{z_0}(z_1)$, but this is irrelevant in the present context. It follows from (5.55) and (5.56) that

$$g(z, z_1) = \ln \left| \frac{1 - \overline{f_{z_0}(z_1)} f_{z_0}(z)}{f_{z_0}(z) - f_{z_0}(z_1)} \right|,$$

and hence

$$g(z_0, z_1) = \ln \frac{1}{|-f_{z_0}(z_1)|},$$

since $f_{z_0}(z_0) = 0$. The proof is now completed by recalling that

$$g(z_1, z_0) = \ln \frac{1}{|f_{z_0}(z_1)|}$$

[cf. (5.53)].

We now turn our attention to the Dirichlet problem for domains more general than a disk:

THEOREM 5.15 (*Dirichlet problem for a Jordan domain*). *Given a Jordan domain G with boundary Γ, let $\mu(z)$ be a continuous real function on Γ, and let $w = f(z)$ be a function mapping G conformally onto the disk $|w| < 1$. Then the Dirichlet problem for G is solved by the function*

$$u(z) = u^*[f(z)] = u^*(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \mu(e^{i\varphi}) \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \varphi)} \, d\varphi. \tag{5.57}$$

Proof. By the boundary correspondence theorem (see p. 164), the function $w = f(z)$ establishes a homeomorphism between Γ and the circle $|w| = 1$, and hence transforms $\mu(z)$ into a function $\mu^*(w)$ which is single-valued and continuous on this circle. The fact that (5.57) solves the Dirichlet problem for G is now an immediate consequence of Theorem 5.8 and the observation that $u(z) = u^*[f(z)]$ is harmonic on G and continuous on $\overline{G}$, with boundary values

$$u(z) = \mu(z) \qquad (z \in G).$$

Remark. We cannot represent (5.57) as an integral along Γ, since in general Γ is not rectifiable. Suppose however that Γ is rectifiable and that the function $w = f(z)$ mapping G onto the disk $|w| < 1$ has a continuous derivative on the *closed* domain $\overline{G} = G \cup \Gamma$.[20] Then (5.57) can be written in the form

$$u(z) = \frac{1}{2\pi} \int_\Gamma \mu(\zeta) \frac{1 - |f(z)|^2}{|1 - f(z)|^2} |f'(\zeta)| \, d\sigma$$

[cf. (5.46)], and in particular,

$$u(z_0) = \frac{1}{2\pi} \int_\Gamma \mu(\zeta) \frac{\partial g(\zeta, z_0)}{\partial \nu} \, d\sigma$$

[cf. (5.54)], where z_0 is the inverse image of the point $w = 0$ under the mapping $w = f(z)$.

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[20] For example, this is the case if Γ is a closed smooth Jordan curve and if the inclination $\alpha(\sigma)$ of the tangent to Γ, regarded as a function of the arc length σ along Γ, satisfies a condition of the form

$$|\alpha(\sigma) - \alpha(\sigma')| \leqslant C|\sigma - \sigma'|^k$$

on Γ, where $C > 0$, $0 < k \leqslant 1$. For the proof of this result, known as *Kellogg's theorem*, see Theorem 6, p. 374 of the book by Goluzin, or Theorem IX.7, p. 361 of the book by Tsuji (both cited in the Bibliography on pp. 326–327).

PROBLEMS

5.1. If u is harmonic, is the same true of u^2? More generally, if u is harmonic, for which functions f is $f(u)$ also harmonic?

5.2. Let $u(x, y)$ and $v(x, y)$ be a pair of conjugate harmonic functions, and let $\chi(u, v)$ be a harmonic function of the variables u and v. Prove that

$$\chi[u(x, y), v(x, y)]$$

is a harmonic function of the variables x and y.

5.3. Calculate the harmonic polynomials $p_n(x, y)$ and $q_n(x, y)$ defined by the equation

$$z^n = p_n + iq_n,$$

where $n = 1, 2, 3, 4$.

5.4. Suppose the integral (5.10) has a given value when evaluated along a piecewise smooth Jordan curve $L \subset G$ joining the points (x_0, y_0) and (x, y). Prove that it has the same value when evaluated along any rectifiable curve $L \subset G$.

Hint. After proving that $f(z)$ is analytic on G, exploit the fact that $f'(z)$ is also analytic on G. Alternatively, use the key lemma of Sec. I.64.

5.5. Given a simply connected domain G and a function $v(x, y)$ harmonic on G, prove that to within an arbitrary real constant, there is a unique analytic function on G with $v(x, y)$ as its imaginary part.

Hint. Cf. Theorem 5.2.

5.6. Find the conjugate harmonic function $v(x, y)$ corresponding to

a) $u(x, y) = x^2 - y^2 + x$ on the domain $|z| < \infty$;

b) $u(x, y) = \dfrac{x}{x^2 + y^2}$ on the domain $0 < |z| < \infty$.

Why is $v(x, y)$ single-valued in case b?

5.7. Let G be the complex plane cut along the negative real axis (including the origin). Then the function

$$u(x, y) = \tfrac{1}{2} \ln (x^2 + y^2)$$

is harmonic on G, and in fact equals $\ln |z|$. Verify by direct use of formula (5.6) that to within an arbitrary real constant, the function

$$v(x, y) = \arg z$$

is the harmonic conjugate of $u(x, y)$.

Hint. Let $(x_0, y_0) = (1, 0)$, and use paths of integration consisting of line segments parallel to the real and imaginary axes. Then use the expressions for $\arg z$ in terms of $\arctan (y/x)$ given on p. I.14.

5.8. Prove that every function $u(x, y)$ which is harmonic on a simply connected domain G can be represented in the form

$$u(x, y) = \ln |f(z)|$$

where $f(z)$ is analytic and nonvanishing on G. What about multiply connected domains? Use this representation to give another proof of Theorem 5.7.

5.9. Prove that if $u(x, y)$ is harmonic on a domain G, then $u(x, y)$ is analytic in the variables x and y, i.e., $u(x, y)$ has a "double power series" expansion of the form

$$u(x, y) = \sum_{m,n=0}^{\infty} c_{mn}(x - x_0)^m(y - y_0)^n$$

in a neighborhood of every point $z_0 = x_0 + iy_0 \in G$.

5.10. Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic at the point $z_0 = x_0 + iy_0$, where $f(z_0) = c_0$. Prove that

a) $f(z) = 2u\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) - \bar{c}_0$;

b) $f(z) = 2iv\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) + \bar{c}_0$.

Hint. Use the result of the preceding problem.

Comment. If $f(z)$ is analytic at the origin, these formulas can be simplified further by setting $z_0 = 0$.

5.11. Find the analytic function $f(z) = u(x, y) + iv(x, y)$, given that

a) $u(x, y) = x^2 - y^2 + 2$; b) $u(x, y) = e^x(x \cos y - y \sin y) - \frac{y}{x^2 + y^2}$;

c) $v(x, y) = x + y - 3$; d) $v(x, y) = \cos x \sinh y - \sinh x \sin y$.

Ans. a) $z^2 + 2 + Ci$ (C real); c) $(1 + i)z - 3i + C$.

5.12. Derive the following version of Schwarz's formula (5.24):

$$f(z) = \frac{1}{\pi i}\int_{|\zeta| = R} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \overline{f(0)}.$$

5.13. Let $u(z)$ be harmonic on the domain $G: |z| > R$, and let $f(z)$ be analytic on G. Derive integral representations analogous to Poisson's integral and Schwarz's formula for $u(z)$ and $f(z)$ in terms of their values on the circle $|z| = R$.

5.14. Let $u(z)$ be harmonic in the upper half-plane G: $\operatorname{Im} z > 0$, and let $f(z)$ be analytic on G. Derive integral representations analogous to Poisson's integral and Schwarz's formula for $u(z)$ and $f(z)$ in terms of their values on the real axis. State conditions for the convergence of the resulting integrals.

Hint. Map G onto the unit disk.

Ans.

$$u(z) = \int_{-\infty}^{\infty} \frac{yu(t)}{(t - x)^2 + y^2}\, dt, \quad f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(t)}{t - z}\, dt + iC \qquad (C \text{ real}).$$

5.15. Let $a_1, a_2, \ldots, a_n$ be points of the real axis such that

$$-\infty < a_1 < a_2 < \cdots < a_n < +\infty.$$

Use the result of Problem 5.14 to find a function $u(z)$ which is harmonic in

the upper half-plane and takes constant values $u_0, u_1, \ldots, u_n$ on the intervals $(-\infty, a_1), (a_1, a_2), \ldots, (a_n, +\infty)$.

Ans.

$$u(z) = \frac{u_0 - u_1}{\pi} \arg(z - a_1) + \frac{u_1 - u_2}{\pi} \arg(z - a_2) + \cdots + \frac{u_{n-1} - u_n}{\pi} \arg(z - a_n) + u_n.$$

5.16. Use the result of Problem 5.15 to deduce the Schwarz-Christoffel transformation (4.14).

Hint. If $w = f(z)$ is the Schwarz-Christoffel transformation, then

$$u(z) = \arg f'(z) = \operatorname{Im} \ln f'(z),$$

where $u(z)$ is the answer to Problem 5.15. Now set $u_{k-1} - u_k = (\alpha_k - 1)\pi$.

5.17. Prove that if $u(z)$ is harmonic and nonnegative on the disk $|z - z_0| \leqslant \rho$, then

$$\frac{\rho - r}{\rho + r} u(z_0) \leqslant u(z) \leqslant \frac{\rho + r}{\rho - r} u(z_0)$$

for every $z = re^{i\theta}$ $(0 \leqslant r < \rho, 0 \leqslant \theta \leqslant 2\pi)$.

Comment. The right-hand inequality has already been encountered in the proof of Harnack's theorem.

5.18. Using the result of the preceding problem, prove the following analogue of Liouville's theorem (Theorem I.16.8): If $u(z)$ is harmonic in the whole finite plane and bounded from below (or above), then $u(z)$ is a constant.

5.19. Prove that on the interval $(-\pi, \pi)$, the Fourier series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cos n\varphi, \qquad -\sum_{n=1}^{\infty} (-1)^{n+1} n \sin n\varphi$$

are summable by Abel's method to $\frac{1}{2}$ and 0, respectively.

Comment. Note that these series can be obtained by formal term-by-term differentiation of the convergent Fourier series

$$\frac{x}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\varphi}{n} \qquad (-\pi < \varphi < \pi).$$

5.20. Prove that if $u(x, y)$ is harmonic on G, so are all the derivatives

$$\frac{\partial^{k+l} u(x, y)}{\partial x^k \, \partial y^l} \qquad (k, l = 1, 2, \ldots). \tag{5.58}$$

5.21. As we know from Theorem 5.11, if the sequence $\{u_n(x, y)\}$ is uniformly convergent on every compact subset of a domain G, and if every term $u_n(x, y)$ is harmonic on G, then the limit function

$$u(x, y) = \lim_{n \to \infty} u_n(x, y)$$

is also harmonic on G. Prove that as $n \to \infty$ each sequence of derivatives

$$\frac{\partial^{k+l} u_n(x, y)}{\partial x^k \, \partial y^l} \qquad (k, l = 1, 2, \ldots)$$

converges uniformly to (5.58) on every compact subset of G. Give the infinite series analogues of this result and Theorem 5.11.

Hint. In the proof, Poisson's integral plays the role of Cauchy's integral in Theorems I.15.6 and I.15.8.

5.22. Prove the uniqueness theorem for conformal mapping (see p. 164).

Hint. Suppose there are two functions $w = f(z)$ and $w = g(z)$, both mapping a given domain G onto the unit disk $|w| < 1$, and satisfying the conditions $f(z_0) = 0$, $f'(z_0) > 0$ and $g(z_0) = 0, g'(z_0) > 0$, where $z_0 \in G$. Then the functions $g[f^{-1}(w)]$ and $f[g^{-1}(w)]$ both map the unit disk onto itself. Use Schwarz's lemma (Theorem I.17.8) to prove that $g[f^{-1}(w)] \equiv w$, i.e., $f(z) \equiv g(z)$.

5.23. Suppose $u(x, y)$ is harmonic on a simply connected domain G, and let L be any closed rectifiable Jordan curve contained in G. Prove that

$$\int_L \frac{\partial u}{\partial \nu} \, ds = 0,$$

where $\partial/\partial\nu$ denotes differentiation along the interior (or exterior) normal to L.

Hint. See Problem I.7.6.

Comment. Concerning the existence of $\partial u/\partial\nu$, see Lemma 2′, p. I.310 and footnote 13, p. I.311.

5.24. Let G be a Jordan domain, and let $u(z)$ be a continuously differentiable real function on G. Then by the *Dirichlet integral* of u (relative to G), we mean the integral

$$D_G(u) = \iint_G \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dx \, dy.$$

When dealing with a fixed domain G, we will often omit the subscript and write just $D(u)$. Prove that

a) $D(u) \geqslant 0$, where $D(u) = 0$ if and only if $u(z)$ is constant on G;

b) $D^2(u, v) \leqslant D(u)D(v)$, where

$$D(u, v) = \iint_G \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx \, dy;$$

c) If $D(u)$ and $D(v)$ are finite, then $D(\lambda u + \mu v)$ is finite for arbitrary real λ and μ, and

$$D(\lambda u + \mu v) = \lambda^2 D(u) + 2\lambda\mu D(u, v) + \mu^2 D(v). \tag{5.59}$$

Write $D(u)$ in polar coordinates. When does equality hold in part b?

5.25. Prove that the Dirichlet integral $D_G(u)$ is a *conformal invariant*, i.e., that if $z^* = f(z)$ maps G onto G^* conformally, and if $u^*(z^*) = u[f^{-1}(z^*)]$, then $D_G(u) = D_{G^*}(u^*)$.

5.26. Let G be a Jordan domain with boundary Γ, and let $\mu(z)$ be a continuous real function on Γ. Let $\mathscr{E}$ be the family of real functions which are continuously differentiable on G and continuous on $\overline{G}$, and which coincide with $\mu(z)$ on Γ. Then, according to Theorem 5.15, $\mathscr{E}$ contains a unique harmonic function $u(z)$. Prove that $D_G(u) \leqslant D_G(h)$ for all $h(z) \in \mathscr{E}$.

Hint. After showing that it is sufficient to consider the case where G is the unit circle $|z| < 1$, let

$$u(z) = \lim_{n \to \infty} u_n(z) \qquad (z = re^{i\theta} \in G),$$

where

$$u_n(z) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos k\theta - \beta_k \sin k\theta) r^k,$$

and α_0, α_k, β_k are given by (5.29). Using Green's theorem in the form[21]

$$\iint_{|z| \leqslant 1} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dx\, dy + \iint_{|z| \leqslant 1} u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) dx\, dy + \int_{|z|=1} u \frac{\partial u}{\partial \nu} ds = 0,$$

where $\partial/\partial\nu$ denotes differentiation along the interior normal, prove that $D(h - u_n, u_n) = 0$. Then use (5.59) to deduce $D(u_n) \leqslant D(h)$. Finally, take the limit as $n \to \infty$ (the result of Problem 5.21 is needed at this point).

5.27. In the preceding problem, prove that if $h(z) \not\equiv u(z)$, then $D(u) < D(h)$, unless $D(h) = \infty$ for all $h \in \mathscr{E}$. Show that the case $D(\infty) = \infty$ for all $h \in \mathscr{E}$ can actually occur.

Hint. Consider the boundary values

$$\mu(\theta) = \sum_{k=1}^{\infty} \frac{\cos k^4\theta}{k^2}$$

on the unit disk.

Comment. The minimization technique for solving the Dirichlet problem implicit in Problems 5.26 and 5.27 is called the *Dirichlet principle.*

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[21] Set

$$P = -u\frac{\partial v}{\partial y}, \qquad Q = u\frac{\partial v}{\partial x}$$

in the more usual form (I.13.2).

6

APPLICATIONS TO FLUID DYNAMICS

28. Irrotational and Solenoidal Flows. The Complex Potential

Consider the motion of an incompressible fluid, i.e., a liquid or gas at velocities much less than the velocity of sound. By a *velocity field* or *flow* (*pattern*), we mean a vector function giving the velocity of the fluid at every point of a given region and at every instant of time. Such a flow is said to be *stationary* if it is independent of time, and *plane-parallel* if it is the same in all planes parallel to a given plane Π and has no components orthogonal to Π. Obviously, in the latter case, there is no loss of generality in assuming that Π is the xy-plane. Thus a stationary plane-parallel flow is characterized by a vector function of two spatial variables x and y, or equivalently, by a complex function

$$w(x, y) = u(x, y) + iv(x, y),$$

where $u(x, y)$ is the x-component of the flow and $v(x, y)$ is its y-component. All the flows considered below are assumed to be stationary and plane-parallel.

Developing this model further, suppose a flow is defined on any domain G (in the xy-plane), with complement $F = G^c$. Then we can regard G as the "flow domain" occupied by the moving fluid, and the closed set F as a collection of impenetrable objects confining the fluid or acting as obstacles to its motion. In fact, each connected component of F (see Problem I.4.12) can be thought of as a separate object, perhaps of infinite size. Strictly speaking, of course, F is the projection onto the xy-plane of a system of

three-dimensional objects.[1] We shall assume that these objects are at rest, but the case where they all have the same constant translational velocity W can be included by the simple device of replacing $w(x, y)$ by $w(x, y) - W$, since there is no conceptual difference between a fluid moving with velocity $-W$ past a fixed set of objects and a set of objects moving with velocity W through a fixed fluid.

Given a flow $w = u + iv$ defined on a domain G, let L be any piecewise smooth Jordan curve of length l contained in G. According to Lemma 1, p. I.309, L has a representation of the form

$$z = \lambda(s) \qquad (0 \leqslant s \leqslant l),$$

where s is the length of the arc of L joining the initial point $\lambda(0)$ to the variable point $\lambda(s)$. Moreover, according to Lemma 2, p. I.310, $\lambda'(s)$ exists and has absolute value 1, except possibly at a finite number of points of the interval $[0, l]$. Clearly,

$$\tau(s) = \lambda'(s) = \frac{dx}{ds} + i\frac{dy}{ds}$$

is the complex number corresponding to the unit vector *tangential* to L at the point $\lambda(s)$, and similarly,

$$\nu(s) = \frac{1}{i}\lambda'(s) = \frac{dy}{ds} - i\frac{dx}{ds}$$

is the complex number corresponding to the unit vector *normal* to L at $\lambda(s)$ (here we have chosen the *exterior normal*).[2] By the component of the flow $w = u + iv$ *tangential* to L at the point $\lambda(s)$, we mean the *scalar product* of the vectors w and $\tau(s)$, i.e., the real vector

$$(u + iv)_\tau = w_\tau = \operatorname{Re} \bar{w}\tau(s) = u\frac{dx}{ds} + v\frac{dy}{ds}.$$

Similarly, by the component of the flow *normal* to L at $\lambda(s)$, we mean the scalar product of the vectors w and $\nu(s)$:

$$(u + iv)_\nu = w_\nu = \operatorname{Re} \bar{w}\nu(s) = -v\frac{dx}{ds} + u\frac{dy}{ds}.$$

(The tangential and normal components may not be defined at a finite number of points in $[0, l]$.) Here our definition of the scalar product is the

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[1] The actual three-dimensional objects should be regarded as very long in the z-direction (ideally, infinitely long). Another approach is to imagine the flow taking place between the plane $z = 0$ and a parallel plane, say $z = 1$, with the planes themselves offering no resistance to the motion of the fluid or to the motion of the objects. The objects are then cylinders of unit height between the planes. See e.g., L. M. Milne-Thomson, *Theoretical Hydrodynamics*, fourth edition, The Macmillan Co., New York (1960), p. 103.

[2] Cf. footnote 13, p. I.311.

natural one. In fact, if $a_1 = \alpha_1 + i\beta_1$ and $a_2 = \alpha_2 + i\beta_2$ are two complex numbers, then, regarded as two-dimensional vectors (α_1, β_1) and (α_2, β_2), they have the scalar product

$$\alpha_1\alpha_2 + \beta_1\beta_2,$$

familiar from elementary vector algebra.[3] But obviously

$$\alpha_1\alpha_2 + \beta_1\beta_2 = \operatorname{Re} \bar{a}_1 a_2.$$

There is also an important vector interpretation of the complex number $\operatorname{Im} \bar{a}_1 a_2$ (see Problem 6.2).

Now let D be any subdomain of the flow domain G, and let $u + iv$ be continuously differentiable on D (i.e., let both u and v be continuously differentiable on D). Then the flow $u + iv$ is said to be *irrotational* in D if the integral

$$\int_L (u + iv)_\tau \, ds = \int_L (u \, dx + v \, dy) \tag{6.1}$$

vanishes for every closed piecewise smooth Jordan curve $L \subset D$. Similarly, the flow $u + iv$ is said to be *solenoidal* in D if the integral

$$\int_L (u + iv)_\nu \, ds = \int_L (-v \, dx + u \, dy) \tag{6.2}$$

vanishes for every closed piecewise smooth Jordan curve $L \subset D$.

If the integral (6.1) is nonzero, it is called the *circulation* around L, while if the integral (6.2) is nonzero, it is called the *flux* through L.[4] Suppose z_0 is a point of G such that $u + iv$ is continuously differentiable in a deleted neighborhood $\mathcal{N}'(z_0)$ and such that the integral (6.1) has a constant nonzero value $\varkappa$ for all $L \subset \mathcal{N}'(z_0)$ for which $z_0 \in I(L)$. Then z_0 is called a *vortex*,[5] and the number $\varkappa$ is called the *strength* of the vortex ("negative strength" is allowed). Similarly, suppose the integral (6.2) has a constant nonzero value m for all $L \subset \mathcal{N}'(z_0)$ for which $z_0 \in I(L)$. Then the point z_0 is called a *source* if $m > 0$ and a *sink* if $m < 0$, and the number $|m|$ is called the *strength* of the source (or sink). A sink can also be regarded as a source of negative strength. The physical meaning of sources, sinks and vortices will emerge in the course of our discussion.

Remark. It is to allow for the possibility of sources, sinks and vortices that we do not insist that $u + iv$ be continuously differentiable at every point

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[3] See e.g., D. V. Widder, *op. cit.*, p. 63.

[4] It is easy to see that the flux through L is just the net amount of fluid flowing outward through L per unit time.

[5] A vortex is often called a *vortex filament*, to suggest the actual physical situation in three dimensions. See e.g., L. M. Milne-Thomson, *op. cit.*, p. 354.

of the flow domain G. In fact, it turns out (see below) that this condition is violated at such "singular points" of the flow.

THEOREM 6.1. *Given a flow $u(x, y) + iv(x, y)$ defined on a domain G, suppose the flow is irrotational and solenoidal in a simply connected subdomain $D \subset G$. Then $u(x, y)$ and $-v(x, y)$ are conjugate harmonic functions on D. Conversely, given a pair of conjugate harmonic functions $u(x, y)$ and $v(x, y)$ on D, the function $u(x, y) - iv(x, y)$ defines an irrotational solenoidal flow in D.*

Proof. As we know from calculus, if

$$\int_L \{P(x, y)\, dx + Q(x, y)\, dy\}$$

vanishes for every piecewise smooth Jordan curve $L \subset D$, where $P(x, y)$ and $Q(x, y)$ are continuously differentiable on D, then

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \tag{6.3}$$

at every point of D, and conversely.[6] Applying (6.3) to the integral (6.1) [recall that u and v are assumed to be continuously differentiable], we obtain the Cauchy-Riemann equation

$$\frac{\partial u}{\partial y} = -\frac{\partial(-v)}{\partial x}, \tag{6.4}$$

and applying (6.3) to the integral (6.2), we obtain the other Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y}. \tag{6.5}$$

Therefore $f(z) = u(x, y) - iv(x, y)$ is analytic on D, and the rest of the proof follows from Theorem 5.1.

Remark 1. The *curl* of the velocity field $w = u + iv$, written curl w, is defined as the vector orthogonal to the xy-plane, with magnitude

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Imagine that the "fluid element" at the point (x, y) suddenly "freezes," and afterwards moves freely with respect to the rest of the fluid. Then it is not hard to show (see Problem 6.3) that the element will rotate with angular velocity

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

[6] See e.g., D. V. Widder, *op. cit.*, Theorem 6, p. 230.

in addition to its translational motion. However, according to (6.4), the curl of the velocity field vanishes at every point of an irrotational flow defined on a simply connected domain D (conversely, if the curl of the velocity field vanishes at every point of a simply connected domain, then the flow is irrotational in D). In other words, the fluid elements in an irrotational flow undergo translational motion with accompanying distortion, but no rotation (hence the word "irrotational"). As shown in Problem 6.4, a vortex is essentially a point at which the curl becomes infinite.

Remark 2. The *divergence* of the velocity field $w = u + iv$, written div w, is defined as the real number

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

According to (6.5), the divergence of the velocity field vanishes at every point of a solenoidal flow defined on a simply connected domain D. Conversely, if the divergence of the velocity field vanishes at every point of a simply connected domain D, then the flow is solenoidal in D. As shown in Problem 6.5, a source or sink is essentially a point at which the divergence becomes infinite.

THEOREM 6.2. *Given a flow $u(x, y) + iv(x, y)$ defined on a domain G, suppose the flow is irrotational and solenoidal in a simply connected subdomain $D \subset G$. Then, to within an arbitrary complex constant, there is a unique single-valued analytic function on D such that*

$$f'(z) = u(x, y) - iv(x, y).$$

Moreover, if

$$f(z) = \varphi(x, y) + i\psi(x, y),$$

then

$$\frac{\partial \varphi}{\partial x} = u, \qquad \frac{\partial \varphi}{\partial y} = v,$$

$$\frac{\partial \psi}{\partial x} = -v, \qquad \frac{\partial \psi}{\partial y} = u,$$

and

$$\varphi(x, y) = \int_{(x_0, y_0)}^{(x, y)} (u\,dx + v\,dy) = \operatorname{Re} \int_{z_0}^{z} f'(\zeta)\,d\zeta,$$

$$\psi(x, y) = \int_{(x_0, y_0)}^{(x, y)} (-v\,dx + u\,dy) = \operatorname{Im} \int_{z_0}^{z} f'(\zeta)\,d\zeta,$$

where each line integral is along any rectifiable curve joining the fixed point $z_0 = (x_0, y_0)$ to the variable point $z = (x, y)$. Conversely, given a single-valued analytic function $f(z)$ on D, the function $\overline{f'(z)}$ defines an irrotational solenoidal flow in D.

Proof. According to Theorem 6.1, the function $f'(z) = u - iv$ is analytic on D. The fact that $f(z)$ is analytic, and the various relations

involving u, v, φ, ψ and f' follow from Theorems I.13.5 and I.13.6, on indefinite integrals of analytic functions. The converse is an immediate consequence of the analyticity of $f'(z)$ on D and Theorem 6.1.

The function $f(z)$ is called the *complex potential* (of the flow), while its real and imaginary parts $\varphi(x, y)$ and $\psi(x, y)$ are called the *velocity potential* and the *stream function*, respectively. The curves

$$\varphi(x, y) = \text{const} \tag{6.6}$$

are called *equipotentials*, while the curves

$$\psi(x, y) = \text{const} \tag{6.7}$$

are called *streamlines*. Since the mapping $\zeta = \xi + i\eta = f(z)$ is conformal on D except at stagnation points and maps (6.6) and (6.7) into the curves

$$\xi = \text{const}, \quad \eta = \text{const},$$

which obviously form an *orthogonal system* (see p. I.134), the two families of curves (6.6) and (6.7) also form an orthogonal system. This conclusion breaks down at *stagnation points*, i.e., at points where $f'(z)$ [and hence the velocity $u + iv$] vanishes.

According to (6.6), the condition

$$\frac{\partial \varphi}{\partial x}\,dx + \frac{\partial \varphi}{\partial y}\,dy = u\,dx + v\,dy = 0$$

characterizes the equipotentials. Thus at every point (x, y) of the flow (except at stagnation points), the velocity is normal to the equipotential through (x, y). Similarly, according to (6.7), the condition

$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = -v\,dx + u\,dy = 0$$

characterizes the streamlines, i.e., at every point (x, y), the velocity is tangential to the streamline through (x, y). Incidentally, this proves once again that the equipotentials and streamlines form an orthogonal system. Moreover, the fact that the velocity vector coincides with the streamlines shows that the streamlines are the actual trajectories of the moving fluid elements.[7]

If the flow domain G contains isolated sources, sinks or vortices then, by deleting these points from G, we obtain a multiply connected domain G' (of course, G itself may be multiply connected). Suppose the flow is irrotational and solenoidal in every simply connected subdomain $D \subset G'$. Then, according to Theorem 6.1, the function $u - iv$, i.e., the complex conjugate of

[7] This conclusion is valid only if the flow is stationary. In the case of a nonstationary flow, where the velocity field is time-dependent, we can still talk about streamlines, i.e., curves tangential to the velocity field at any given instant of time, but now the streamlines do not in general coincide with the trajectories of the fluid elements.

the velocity $u + iv$ is a single-valued analytic function on G'. Although the complex potential

$$f(z) = \int_{z_0}^{z} (u - iv)\, d\zeta = \varphi(x, y) + i\psi(x, y)$$

is now in general multiple-valued, it has single-valued analytic branches on every simply connected subdomain $D \subset G'$, and every branch has the same derivative $u - iv$ (see Theorem I.13.7), i.e., two different branches of $f(z)$ can only differ by a constant. The sources, sinks or vortices excluded from G correspond to singular points (of single-valued or multiple-valued character) of the complex potential. Conversely, given any multiple-valued function $f(z)$ with isolated singular points in a domain G, suppose $f(z)$ has single-valued analytic branches on every simply connected subdomain $D \subset G'$ (where G' is the domain obtained by excluding the singular points from G), and suppose $f'(z)$ is single-valued on G'. Then $f(z)$ can be regarded as the complex potential of a flow in G, which is irrotational and solenoidal in every simply connected subdomain $D \subset G'$, and the singular points can be interpreted in terms of sources, sinks and vortices of the flow. These remarks will be amply illustrated by the examples given below.

Finally, we should not overlook an extra restriction that must be satisfied by any physical flow: The surface of each object confining the flow, i.e., each connected component of the boundary G^b of the flow domain G, must be part of a streamline, since the flow can have no component normal to such a surface.[8] Correspondingly, if $f(z)$ is to be the complex potential of a flow, then $\psi(x, y) = \operatorname{Im} f(z)$ must be constant on each component of G^b.

29. Examples

We now illustrate the theory developed in the preceding section, beginning with some cases where the complex potential has no singular points.

Example 1. The entire linear function

$$f(z) = az \tag{6.8}$$

can be regarded as the complex potential of a flow occupying the whole plane, with uniform velocity

$$\overline{f'(z)} = \bar{a}$$

at every point. Writing $a = \alpha + i\beta$, we obtain the velocity potential

$$\varphi(x, y) = \alpha x - \beta y$$

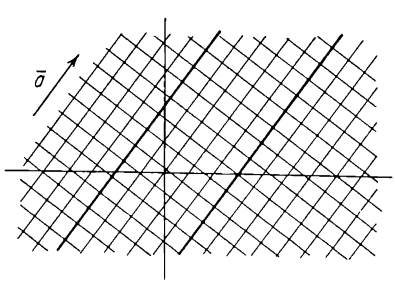

FIGURE 6.1

[8] See e.g., L. M. Milne-Thomson, *op. cit.*, p. 77. Recall that G^b actually represents the projection onto the xy-plane of a set of cylindrical objects.

and the stream function

$$\psi(x, y) = \beta x + \alpha y.$$

The corresponding orthogonal system of equipotentials and streamlines is shown in Figure 6.1. The same function (6.8) represents the complex potential for uniform flow in a strip whose boundary consists of two lines parallel to the vector $\bar{a}$ (like the two heavy lines shown in the figure).

Example 2. The complex potential

$$f(z) = z^2 \tag{6.9}$$

also describes a flow occupying the whole plane, but this time the flow is nonuniform and the fluid particle at the point z has velocity

$$\overline{f'(z)} = 2\bar{z}.$$

For the corresponding velocity potential and stream function we have

$$\varphi(x, y) = x^2 - y^2$$

and

$$\psi(x, y) = 2xy.$$

The equipotentials

$$x^2 - y^2 = \text{const}$$

and streamlines

$$2xy = \text{const}$$

are shown in Figure 6.2, and consist of two families of equilateral hyperbolas.

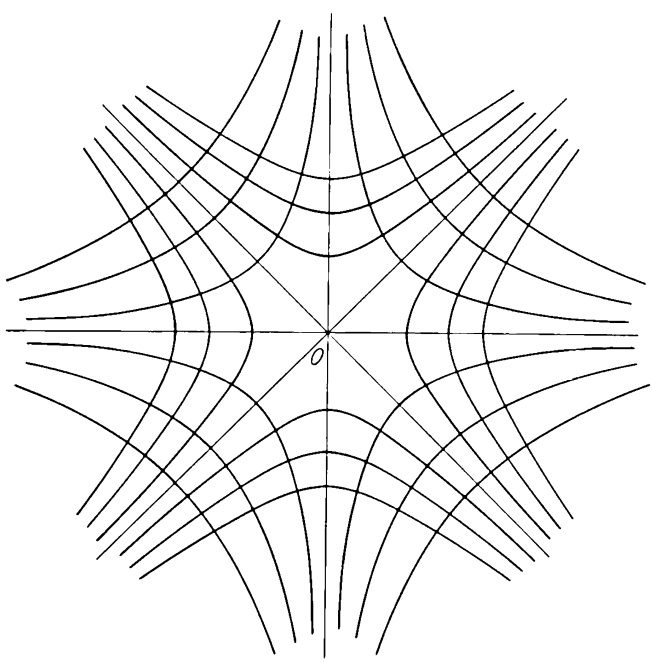

FIGURE 6.2

The coordinate axes are also streamlines ($2xy = 0$), and intersect in a stagnation point at the origin. It is clear that the same function (6.9) represents the complex potential of a flow in any quadrant of the xy-plane, where the sides of the quadrant correspond to the walls of a vessel containing the moving fluid.

Example 3. By using conformal mapping, we can study the motion of a fluid with velocity

$$W = U + iV = |W|e^{i\alpha}$$

at infinity, flowing past a circular cylinder. Let $|z| = R$ be the projection of the cylinder onto the xy-plane (i.e., its intersection with the xy-plane). Then the flow domain G is the exterior of the circle $|z| = R$. The function

$$z_1 = \frac{e^{-i\alpha} z}{R}$$

is a conformal mapping of the exterior of the circle $|z| = R$ onto the exterior of the unit circle in the z_1-plane, which carries the vector $|W|e^{i\alpha}$ into the vector of length $|W|/R$ directed along the positive real axis, while

$$z_2 = \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right)$$

is a conformal mapping of the exterior of the unit circle onto the exterior of the segment $-1 \leqslant x_2 \leqslant 1$, $y_2 = 0$ of the real axis (see Sec. I.51). Therefore the function

$$z_2 = \frac{1}{2}\left(\frac{e^{-i\alpha} z}{R} + \frac{R}{e^{-i\alpha} z}\right) \tag{6.10}$$

is analytic on G, and moreover, as required, its imaginary part y_2 has a constant value on the boundary of G (in fact, $y_2 = 0$ for $|z| = R$). It follows that if the function (6.10) is regarded as the complex potential of a flow in G, then the boundary of G is a streamline, i.e., (6.10) corresponds to flow past the cylinder $|z| = R$.

For the velocity of the flow, we have

$$\overline{\frac{dz_2}{dz}} = \frac{1}{2}\overline{\left(\frac{e^{-i\alpha}}{R} - \frac{R}{e^{-i\alpha} z^2}\right)} = \frac{1}{2}\left(\frac{e^{i\alpha}}{R} - \frac{Re^{-i\alpha}}{\bar{z}^2}\right),$$

and hence the velocity at infinity equals

$$\frac{1}{2}\frac{e^{i\alpha}}{R},$$

which differs from $|W|e^{i\alpha}$ only by the positive real factor $1/2|W|R$. Thus, multiplying (6.10) by $2|W|R$, we finally obtain the appropriate complex potential

$$f(z) = |W|e^{-i\alpha} z + \frac{|W|e^{i\alpha} R^2}{z}$$

or

$$f(z) = (U - iV)z + \frac{(U + iV)R^2}{z}.$$

As before, Im $f(z) = 0$ on the circle $|z| = R$, while

$$\lim_{z \to \infty} f'(z) = U - iV = \overline{U + iV}.$$

For the corresponding velocity potential and stream function, we have

$$\varphi(x, y) = \text{Re } f(z) = (Ux + Vy)\left(1 - \frac{R^2}{x^2 + y^2}\right)$$

and

$$\psi(x, y) = \text{Im } f(z) = (-Vx + Uy)\left(1 - \frac{R^2}{x^2 + y^2}\right).$$

The equipotentials

$$(Ux + Vy)(x^2 + y^2 + R^2) = C_1(x^2 + y^2)$$

and streamlines

$$(-Vx + Uy)(x^2 + y^2 - R^2) = C_2(x^2 + y^2)$$

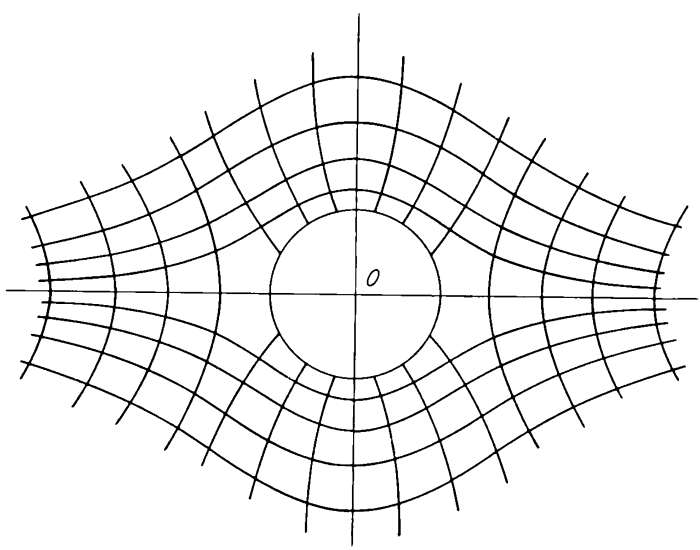

FIGURE 6.3

are both families of algebraic curves of degree 3, and are shown in Figure 6.3 (corresponding to the case $\alpha = 0$). It should be noted that for $C_2 = 0$ the streamline reduces to the circle $x^2 + y^2 = R^2$, together with the straight line

$$-Vx + Uy = 0$$

passing through the origin parallel to the vector $U + iV$ representing the velocity at infinity. At the points $\pm Re^{i\alpha}$ where these curves intersect, the velocity

$$f'(z) = |W|e^{i\alpha} - \frac{|W|e^{-i\alpha} R^2}{\bar{z}^2}$$

vanishes. Thus the points $\pm Re^{i\alpha}$ are stagnation points, but there are no others. In Sec. 30 we shall see that other flows past the same cylinder are possible, with a different complex potential.

Next we allow the complex potential to have singular points, beginning with the case of logarithmic branch points.

Example 4. The multiple-valued function

$$f(z) = \operatorname{Ln} z,$$

defined on the domain $0 < |z| < \infty$, has a single-valued derivative

$$f'(z) = \frac{1}{z},$$

and hence can be regarded as the complex potential of a stationary plane-parallel flow, with velocity potential

$$\varphi(x, y) = \ln |z|$$

and stream function

$$\psi(x, y) = \operatorname{Arg} z.$$

Each equipotential

$$\ln |z| = \text{const} \quad \text{or} \quad |z| = \text{const}$$

is a circle with its center at the origin, while each streamline

$$\operatorname{Arg} z = \text{const}$$

is a ray emanating from the origin. The velocity

$$\overline{f'(z)} = \frac{1}{\bar{z}} = \frac{z}{|z|^2}$$

at the point z is directed along the ray $\operatorname{Arg} z = \text{const}$ joining the origin and the point at infinity and passing through z. Thus fluid elements move from the origin to the point at infinity with a velocity that is very large near the origin and very small far from the origin ($|\overline{f'(z)}| = 1/|z|$). This flow pattern suggests that we regard one of the branch points of $\operatorname{Ln} z$, i.e., the point $z = 0$ as a source, and the other branch point $z = \infty$ as a sink. In fact, let L be any circle with its center at the origin. Then, since

$$\int_L (-v\,dx + u\,dy) = \operatorname{Im} \int_L f'(z)\,dz = \operatorname{Im} \int_L \frac{dz}{z} = 2\pi,$$

we see that both the source at the origin and the sink at infinity are of strength 2π (see p. 176).[9] Roughly speaking, the fluid emanates from the source with infinite velocity and disappears at the point $z = \infty$ with zero velocity.

Example 5. Consider the somewhat different complex potential

$$f(z) = -i \operatorname{Ln} z,$$

with velocity potential

$$\varphi(x, y) = \operatorname{Arg} z$$

and stream function

$$\psi(x, y) = -\ln |z|.$$

Now the equipotentials are rays emanating from the origin, and the streamlines are circles with their centers at the origin, i.e., the only difference between this case and Example 4 is that the roles of the functions φ and ψ have been interchanged. The velocity at the point z equals

$$\overline{|f'(z)|} = \frac{i}{\bar{z}} = \frac{iz}{|z|^2},$$

and is directed (in the positive direction) along the tangent to the circle through z with its center at the origin, i.e., every fluid element moves in the counterclockwise direction along a circle with its center at the origin. As before, the velocity is very large near the origin, and very small far from the origin. The time required to traverse the circle $|z| = r$ equals

$$\frac{2\pi r}{1/r} = 2\pi r^2,$$

and grows like the square of the radius. It is easy to see that there are no sources and sinks, since the stream function is single-valued. On the other hand, there is now a vortex of strength 2π at the origin and a vortex of strength -2π at infinity, since

$$\int_L (u\,dx + v\,dy) = \operatorname{Re} \int_L f'(z)\,dz = \operatorname{Re} \int_L \frac{-i\,dz}{z} = 2\pi,$$

where L is any circle with center at the origin.

Example 6. Generalizing the preceding two examples, we see at once that the complex potential

$$f(z) = \frac{m}{2\pi} \operatorname{Ln} \frac{z - a}{z - b}$$

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[9] In physical terms, this means that the net amount of fluid flowing outward through L per unit time (more precisely, flowing through the surface of a right circular cylinder of unit height with base L) equals 2π.

corresponds to a flow with a source of strength m at the point a and a sink of strength m at the point b. Similarly,

$$f(z) = \frac{\varkappa}{2\pi i} \operatorname{Ln} \frac{z - a}{z - b}$$

corresponds to a flow with a pair of vortices of strength $\pm\varkappa$ at the points a and b (the circulations of the vortices have opposite signs). The flow pattern for both cases is shown in Figure 6.4. In the first case, the streamlines are

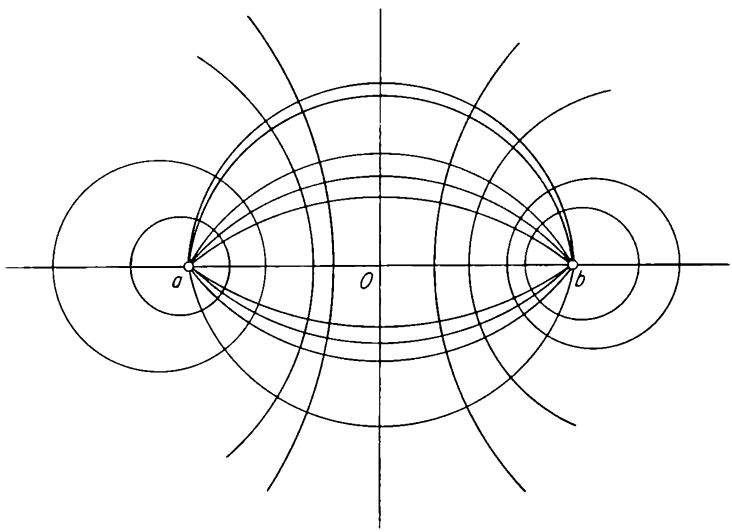

FIGURE 6.4

arcs of circles joining the points a and b, while the equipotentials are the circles orthogonal to these arcs. In the second case, the roles of the streamlines and the equipotentials are interchanged.

Example 7. We can also consider the complex potential

$$f(z) = \frac{m - i\varkappa}{2\pi} \operatorname{Ln} \frac{z - a}{z - b},$$

corresponding to the case where each of the points a and b is both a source (or sink) of strength m and a vortex of strength $\varkappa$. Here we have the velocity potential

$$\varphi(x, y) = \frac{m}{2\pi} \ln \frac{|z - a|}{|z - b|} + \frac{\varkappa}{2\pi} \operatorname{Arg} \frac{z - a}{z - b}$$

and the stream function

$$\psi(x, y) = \frac{m}{2\pi} \operatorname{Arg} \frac{z - a}{z - b} - \frac{\varkappa}{2\pi} \ln \frac{|z - a|}{|z - b|}.$$

The equipotentials and streamlines consist of two orthogonal families of *double logarithmic spirals*, winding about the points a and b.[10] Figure 6.5 shows three curves from each family. A point like a (or b), where a source and vortex are combined, is called a *spiral vortex*, and the complex number $m - ix$ is called the *strength* of the spiral vortex.

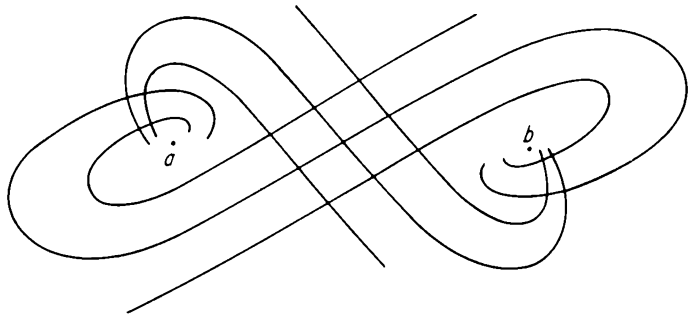

FIGURE 6.5

Example 8. Finally we give a fluid-dynamical interpretation of a pole of an analytic function. Let

$$f(z) = \frac{m - ix}{2\pi} \operatorname{Ln} \frac{z - a}{z - b}$$

be the complex potential corresponding to two spiral vortices, at the points a and b. We write $f(z)$ in the form

$$f(z) = \frac{(m - ix)(b - a)}{2\pi} \frac{\operatorname{Ln}(z - a) - \operatorname{Ln}(z - b)}{b - a},$$

and then let b approach a and $m - ix$ approach infinity in such a way that the product

$$(m - ix)(b - a)$$

has a finite nonzero limit. As a result of this passage to the limit, we obtain the function

$$F(z) = \frac{Re^{i\alpha}}{2\pi} \frac{1}{z - a},$$

which has just one singular point, i.e., a simple pole at the point a. Thus simple poles can be obtained by merging two spiral vortices of the same

[10] Since these double logarithmic spirals are transformed into ordinary logarithmic spirals (see p. I.143) by the Möbius transformation

$$\zeta = \frac{z - a}{z - b},$$

all the curves in each family intersect any circular arc joining the points a and b at the same angle.

infinite strength. Naturally, we can obtain the same result starting from a source and a sink ($\varkappa = 0$), or from two vortices ($m = 0$).

Setting $z - a = re^{i\theta}$, we write the velocity potential and stream function in polar coordinates:

$$\varphi(r, \theta) = \frac{R}{2\pi r} \cos(\alpha - \theta), \qquad \psi(r, \theta) = \frac{R}{2\pi r} \sin(\alpha - \theta).$$

Then the equipotentials and streamlines are

$$\frac{R}{2\pi r} \cos(\alpha - \theta) = \text{const}, \qquad \frac{R}{2\pi r} \sin(\alpha - \theta) = \text{const},$$

i.e., two orthogonal families of circles

$$r = C_1 \cos(\alpha - \theta), \qquad r = C_2 \sin(\alpha - \theta),$$

where C_1 and C_2 are constants. As shown in Figure 6.6, the circles of the first family (the equipotentials) have the common tangent vector $iRe^{i\alpha}$ at the point a, while the circles of the second family (the streamlines) have the common tangent vector $Re^{i\alpha}$ at the same point.

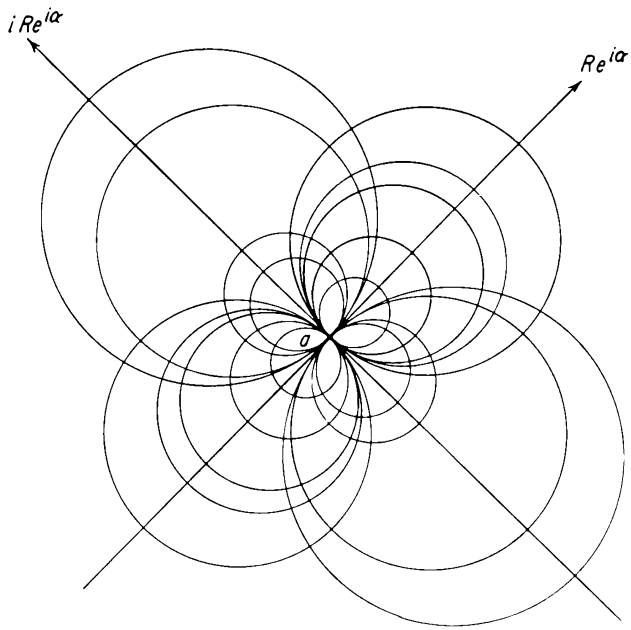

FIGURE 6.6

In the same way, we can construct a pole of the second order by merging two poles of the first order, a pole of the third order by merging two poles of the second order, and so on.

30. Flow past a Circular Cylinder

We now return to the problem of flow past a circular cylinder, deriving a result of a more general character than that found in Example 3, p. 182. Thus suppose it is required to find the complex potential $f(z)$ of a flow in the domain $|z| > R$, given that the velocity at infinity equals $U + iV$ and that there are no sources, sinks or vortices in the domain $R < |z| < \infty$. Then the derivative $f'(z)$, which is the complex conjugate of the velocity at the point z, must be a single-valued analytic function on the domain $|z| > R$, taking the value $U - iV$ at infinity. Therefore, since infinity must be a regular point of $f'(z)$, we have

$$f'(z) = U - iV + \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \cdots,$$

which implies

$$f(z) = (U - iV)z + A_1 \operatorname{Ln} z - \frac{A_2}{z} - \frac{A_3}{2z^2} - \cdots, \tag{6.11}$$

where the constant of integration has been temporarily set equal to zero. To find the corresponding stream function $\psi(r, \theta) = \operatorname{Im} f(z)$ in polar coordinates, we write

$$z = re^{i\theta}, \quad A_1 = a_1 + ib_1, \quad A_2 = a_2 + ib_2, \quad A_3 = a_3 + ib_3, \ldots$$

This leads to the expansion

$$\begin{aligned}\psi(r, \theta) &= Ur \sin \theta - Vr \cos \theta + a_1\theta + b_1 \ln r + \frac{a_2}{r} \sin \theta - \frac{b_2}{r} \cos \theta \\ &\qquad + \frac{a_3}{2r^2} \sin 2\theta - \frac{b_3}{2r^2} \cos 2\theta + \cdots \\ &= a_1\theta + b_1 \ln r + \frac{a_2 + Ur^2}{r} \sin \theta - \frac{b_2 + Vr^2}{r} \cos \theta \\ &\qquad + \frac{a_3}{2r^2} \sin 2\theta - \frac{b_3}{2r^2} \cos 2\theta + \cdots,\end{aligned} \tag{6.12}$$

which converges for $r > R$.

Since the circle $|z| = R$ has to be one of the streamlines, the function $\psi(r, \theta)$ must equal a constant for $r = R$ and arbitrary θ. This condition will be satisfied if we make the following choice of coefficients in (6.12):

$$a_1 = 0, \quad a_2 + UR^2 = 0, \quad b_2 + VR^2 = 0, \quad a_3 = 0, \quad b_3 = 0, \ldots$$

Then (6.11) reduces to

$$f(z) = ib_1 \operatorname{Ln} z + (U - iV)z + \frac{(U + iV)R^2}{z}, \tag{6.13}$$

where b_1 is an arbitrary real number. It follows that

$$f'(z) = \frac{ib_1}{z} + U - iV - \frac{(U + iV)R^2}{z^2},$$

and hence

$$\int_L f'(z)\,dz = \int_L \frac{ib_1}{z}\,dz = -2\pi b_1,$$

where L is any circle $|z| = r > R$. Therefore the flux through L vanishes, while the circulation around L equals $-2\pi b_1$. To obtain a given value $\varkappa$ of the circulation, we need only set

$$-2\pi b_1 = \varkappa \quad \text{or} \quad b_1 = -\frac{\varkappa}{2\pi},$$

so that (6.13) takes the final form

$$f(z) = \frac{\varkappa}{2\pi i}\operatorname{Ln} z + (U - iV)z + \frac{(U + iV)R^2}{z} + \text{const}, \tag{6.14}$$

where we have introduced an extra constant (which has no effect on the velocity field). Thus the complex potential is the superposition of the flow without circulation found in Example 3, p. 182, and the "purely circulatory flow" of the type described in Example 5, p. 185,[11] corresponding to vortices of strength $\pm\varkappa$ at the origin and at infinity.

Next we show that (6.14) is the most general solution of the problem of flow past a circular cylinder with a given circulation $\varkappa$ and a given velocity $U + iV$ at infinity. To see this, let $f_1(z)$ be another complex potential satisfying the same conditions. Then the difference

$$f_1'(z) - f'(z),$$

which is the complex conjugate of the difference between the corresponding velocity fields, is single-valued and analytic on the domain $|z| > R$, and moreover vanishes at infinity. Therefore

$$f_1'(z) - f'(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots, \tag{6.15}$$

and hence

$$\int_L [f_1'(z) - f'(z)]\,dz = c_1 \int_L \frac{dz}{z} = 2\pi i c_1.$$

Since, by hypothesis, the circulation around L is the same for both flows,

$$\operatorname{Re} \int_L [f_1'(z) - f'(z)]\,dz = -2\pi \operatorname{Im} c_1 = 0,$$

i.e., c_1 must be a real number. Integrating (6.15), we find that the difference between the complex potentials themselves is

$$f_1(z) - f(z) = c_0 + c_1 \operatorname{Ln} z - \frac{c_2}{z} - \frac{c_3}{2z^2} - \cdots,$$

---

[11] Note that the flow corresponding to (6.14) is still irrotational in every simply connected subdomain of the flow domain $|z| > R$. Circulation is present only when the origin lies inside the circle L.

while the difference between their imaginary parts (i.e., the difference between the stream functions) is

$$\begin{aligned}\delta(r, \theta) &= \operatorname{Im}[f_1(z) - f(z)] \\ &= \beta_0 + c_1\theta + \frac{\alpha_2}{r}\sin\theta - \frac{\beta_2}{r}\cos\theta + \frac{\alpha_3}{2r^2}\sin 2\theta - \frac{\beta_3}{2r^2}\cos 2\theta + \cdots,\end{aligned} \tag{6.16}$$

where we have written $c_k = \alpha_k + i\beta_k$ $(k = 0, 2, 3, \ldots)$. Since the circle $|z| = R$ must be a streamline for both flows, we have

$$\delta(R, \theta) = \text{const} = c.$$

But according to (6.16), the function $\delta(r, \theta) - c_1\theta$ is a single-valued function of r and θ, and hence

$$\delta(R, \theta) - c_1\theta = c - c_1\theta$$

must be a single-valued function of θ, which implies that $c_1 = 0$. Thus we have the expansion

$$\delta(r, \theta) = \beta_0 + \frac{\alpha_2}{r}\sin\theta - \frac{\beta_2}{r}\cos\theta + \frac{\alpha_3}{2r^2}\sin 2\theta - \frac{\beta_3}{2r^2}\cos 2\theta + \cdots,$$

from which it is apparent that $\delta(r, \theta)$ is a single-valued harmonic function on the domain $r > R$, satisfying the boundary condition $\delta(R, \theta) = c$. It follows from Theorem 5.7, Corollary 2 that $\delta(r, \theta) \equiv c$, and hence the analytic function $f_1(z) - f(z)$ with imaginary part $\delta(r, \theta)$ is also a constant (why?). Therefore

$$f_1(z) = f(z) + \text{const} = \frac{\varkappa}{2\pi i}\operatorname{Ln} z + (U - iV)z + \frac{(U + iV)R^2}{z} + \text{const},$$

i.e., $f_1(z)$ coincides with (6.14), as asserted.

The velocity potential and stream function associated with the complex potential (6.14) are

$$\begin{aligned}\varphi(x, y) &= \frac{\varkappa}{2\pi}\operatorname{Arg} z + (Ux + Vy)\left(1 + \frac{R^2}{x^2 + y^2}\right) + \text{const}, \\ \psi(x, y) &= -\frac{\varkappa}{2\pi}\ln|z| + (-Vx + Uy)\left(1 - \frac{R^2}{x^2 + y^2}\right) + \text{const},\end{aligned}$$

with corresponding equipotentials and streamlines

$$\begin{aligned}\frac{\varkappa}{2\pi}\operatorname{Arg} z + (Ux + Vy)\left(1 + \frac{R^2}{x^2 + y^2}\right) &= \text{const}, \\ -\frac{\varkappa}{2\pi}\ln|z| + (-Vx + Uy)\left(1 + \frac{R^2}{x^2 + y^2}\right) &= \text{const},\end{aligned}$$

which have already been studied in Example 3, p. 182 for the case of no circulation ($\varkappa = 0$). When circulation is present ($\varkappa \neq 0$), the equipotentials

and streamlines are transcendental curves whose shape depends on the relation between $\varkappa$ and $U - iV$. Assuming for simplicity that $V = 0$ (we can always arrive at this case by making a preliminary rotation of the coordinate axes), we now look for the stagnation points of the flow, i.e., the points at which the velocity vanishes. Since

$$\overline{f'(z)} = -\frac{\varkappa}{2\pi i}\frac{1}{\bar{z}} + U - \frac{UR^2}{\bar{z}^2},$$

these points satisfy the quadratic equation

$$\bar{z}^2 - \frac{\varkappa}{2\pi i U}\bar{z} - R^2 = 0$$

or

$$z^2 + \frac{\varkappa}{2\pi i U}z - R^2 = 0,$$

with solutions

$$\frac{i}{4\pi U} \pm \sqrt{R^2 - \left(\frac{\varkappa}{4\pi U}\right)^2}. \tag{6.17}$$

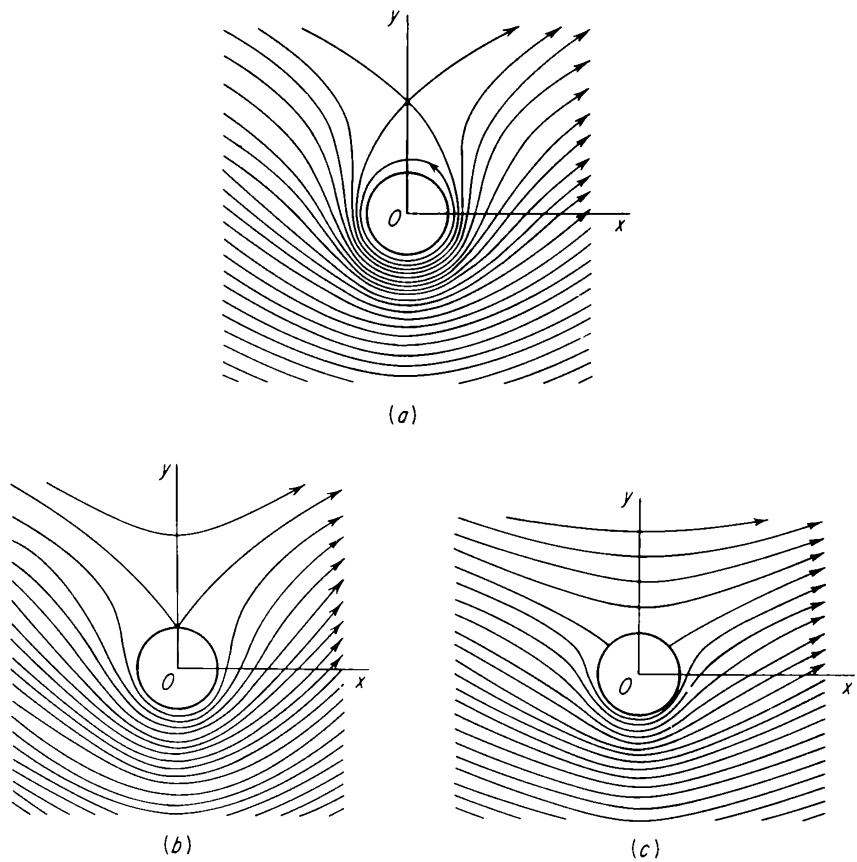

FIGURE 6.7

If $|\varkappa| > 4\pi|U|$, both stagnation points z_1 and z_2 are purely imaginary, but, because of the relation $z_1 z_2 = -R^2$, only one of them lies outside the circle $|z| = R$, i.e., in the domain occupied by the moving fluid. The streamlines for this case are shown in Figure 6.7(a). If $|\varkappa| = 4\pi R|U|$, there is only one stagnation point, at the point where the imaginary axis intersects the circle $|z| = R$ [see Figure 6.7(b)]. Finally, if $|\varkappa| > 4\pi R|U|$, then, just as in the case of flow without circulation, there are two stagnation points (6.17) on the circle $|z| = R$, symmetrically located with respect to the imaginary axis [see Figure 6.7(c)].

*31. Flow past an Arbitrary Cylindrical Object. The Kutta-Joukowski Theorem

In the preceding section, we solved the problem of stationary plane-parallel flow past a circle (the projection of a circular cylinder onto the complex plane). With this problem as our starting point, we can use conformal mapping to solve the problem of flow past an arbitrary cylindrical object. Thus let L be a closed Jordan curve in the z-plane, and suppose it is required to find the complex potential of a flow past L with a given velocity $W = U + iV$ at infinity. We map the exterior of L conformally onto the exterior of the unit circle $\Gamma: |\omega| = 1$ in the ω-plane, in such a way that the point $z = \infty$ goes into the point $\omega = \infty$ (recall that a univalent function is allowed to have a pole). Let $\omega = F(z)$ be the function which accomplishes this mapping. Then in a neighborhood of the point $z = \infty$, $F(z)$ has the expansion

$$F(z) = a_1 z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots,$$

where $a_{-1} \neq 0$ (why are there no higher powers of z?). To be explicit, we impose the extra requirement that the coefficient a_1 be a positive real number, i.e., that $F'(\infty) > 0$.[12] These conditions uniquely determine $F(z)$, and in fact $F(z)$ establishes a homeomorphism between $\overline{E(L)}$ and $\overline{E(\Gamma)}$.[13]

---

[12] In the present case, we *define*

$$F'(\infty) = \lim_{z \to \infty} F'(z),$$

although $F(z)$ is not analytic at ∞. In the case where $F(z)$ is analytic at ∞, $F'(\infty)$ is defined differently (see p. I.125 and Problem I.8.10). In any event, the condition $F'(\infty) \neq 0 (\neq \infty)$ guarantees that $F(z)$ is conformal at ∞ (prove this, recalling Problem I.8.9).

[13] To see this, apply the theorems cited on p. 164, after using the preliminary transformations

$$\tilde{z} = \chi(z) = \frac{1}{z - z_0}, \qquad z_0 \in E(L), \qquad \tilde{\omega} = \frac{1}{\omega}$$

to reduce the problem to the mapping of the interior of $\chi(L)$ onto the interior of the unit circle.

Under the mapping $\omega = F(z)$, the complex potential $f(z)$ for flow past the contour L goes into the complex potential $f^*(\omega)$ for flow past the unit circle, and therefore is of the form

$$f(z) = f[F^{-1}(\omega)] = f^*(\omega) = \frac{\varkappa}{2\pi i} \operatorname{Ln} \omega + (u_\infty - iv_\infty)\omega + \frac{u_\infty + iv_\infty}{\omega} + \text{const}$$

[cf. (6.14)], where

$$u_\infty - iv_\infty = f^{*\prime}(\infty) = \frac{f'(\infty)}{F'(\infty)} = \frac{U - iV}{F'(\infty)}.$$

It follows that

$$f(z) = \frac{\varkappa}{2\pi i} \operatorname{Ln} F(z) + \frac{U - iV}{F'(\infty)} F(z) + \frac{U + iV}{F'(\infty)F(z)} + \text{const}. \tag{6.18}$$

The additive complex constant appearing in this formula has no physical significance, but the real constant $\varkappa$ represents the circulation around any closed Jordan curve containing L in its interior, e.g., around the curve γ_r which is the inverse image of the circle $\Gamma_r: |\omega| = r > 1$ under the mapping $\omega = F(z)$. In fact, we have

$$\int_{\gamma_r} f'(z)\, dz = \operatorname*{Var}_{z \in \gamma_r} f(z) = \frac{\varkappa}{2\pi} \operatorname*{Var}_{z \in \gamma_r} \operatorname{Arg} F(z).$$

But as z traverses γ_r once in the positive direction, $\omega = F(z)$ traverses Γ_r once in the positive direction,[14] i.e.,

$$\operatorname*{Var}_{z \in \gamma_r} \operatorname{Arg} F(z) = 2\pi,$$

and hence

$$\int_{\gamma_r} f'(z)\, dz = \varkappa,$$

as asserted. Thus (6.18) corresponds to a flow past the contour L, with circulation $\varkappa$ and velocity $U + iV$ at infinity, where $F(z)$ is the function mapping $E(L)$ conformally onto $E(\Gamma)$ and satisfying the conditions $F(\infty) = \infty$, $F'(\infty) > 0$.

We now use (6.18) to find the complex potential of the flow past the profile of an airplane wing. To obtain such a profile, consider two circles γ and γ' in the ζ-plane, where $\gamma \subset I(\gamma')$, such that γ passes through the points $\zeta = \pm 1$ and is tangent to γ' at the point $\zeta = 1$ (see Figure 6.8). Then the Joukowksi function

$$z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \tag{6.19}$$

transforms the circle γ into a circular arc δ with end points $z = \pm 1$, while mapping $E(\gamma)$ conformally onto the domain D with boundary δ (see p. I.201,

[14] Cf. the proof of Theorem 4.5.

esp. Figure I.10.11 and Remark 1). Thus (6.19) is one-to-one on γ', which it maps into a closed curve $\delta' \subset D \cup \{1\}$ (note that δ and δ' share the single point $z = 1$). Since the derivative

$$\frac{dz}{d\zeta} = \frac{1}{2}\left(1 - \frac{1}{\zeta^2}\right) = \frac{\zeta^2 - 1}{2\zeta^2}$$

has a simple zero at $\zeta = 1$, any angle with its vertex at $\zeta = 1$ is mapped by (6.19) into an angle of twice the size with its vertex at $z = 1$. But the angle between γ and γ' at $\zeta = 1$ is zero, and hence the same is true of the angle between δ and δ' at $z = 1$. The curve δ', known as the *Joukowski profile*, is wing-shaped, with a blunt leading edge and a sharp trailing edge, as shown in Figure 6.8. Its size and shape can be varied by varying the circles γ and γ', and by making similarity transformations.

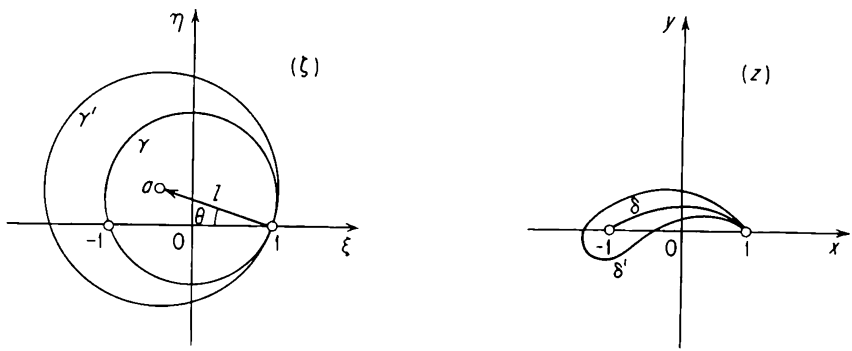

FIGURE 6.8

We must still find the function $\omega = F(z)$ mapping the domain $E(\delta')$ conformally onto $E(\Gamma)$. First we note that (6.19) maps $E(\gamma')$ conformally onto $E(\delta')$ and carries the point $\zeta = \infty$ into the point $z = \infty$, while

$$\left.\frac{dz}{d\zeta}\right|_{\zeta=\infty} = \frac{1}{2}.$$

But if the circle γ' has center a and radius l, the function

$$\omega = \frac{1}{l}(\zeta - a)$$

maps $E(\gamma')$ conformally onto $E(\Gamma)$. Therefore the function

$$z = \tfrac{1}{2}[(l\omega + a) + (l\omega + a)^{-1}] \tag{6.20}$$

maps $E(\Gamma)$ conformally onto $E(\delta')$ and carries the point $\omega = \infty$ into the point $z = \infty$, while

$$\left.\frac{dz}{d\omega}\right|_{\omega=\infty} = \frac{l}{2}.$$

The desired function $\omega = F(z)$ is the inverse of (6.20), i.e.,

$$\omega = F(z) = \frac{1}{l}(-a + z + \sqrt{z^2 - 1}), \tag{6.21}$$

where we must choose the branch of the square root which makes $F(\infty) = \infty$. This choice gives

$$F'(\infty) = \frac{2}{l} > 0,$$

as required. Using (6.21), we can write the complex potential as

$$f(z) = \frac{\varkappa}{2\pi i} \operatorname{Ln}\left[\frac{1}{l}(-a + z + \sqrt{z^2 - 1})\right] + \frac{U - iV}{2}(-a + z + \sqrt{z^2 - 1}) + \frac{l^2(U + iV)}{2(-a + z + \sqrt{z^2 - 1})} + \text{const.} \tag{6.22}$$

Differentiating (6.22), we find that

$$f'(z) = \left[\frac{\varkappa}{2\pi i}\frac{1}{-a + z + \sqrt{z^2 - 1}} + \frac{U - iV}{2} - \frac{l^2(U + iV)}{2(-a + z + \sqrt{z^2 - 1})^2}\right] \times \left(1 + \frac{z}{\sqrt{z^2 - 1}}\right).$$

Therefore, in order for the derivative $f'(z)$ and hence the velocity to be finite near the trailing edge of the wing, i.e., near the point $z = 1$, we have to impose the following condition relating the circulation $\varkappa$, the velocity $U + iV$ at infinity and the parameters a, l determining the shape of the wing:

$$\frac{\varkappa}{2\pi i}\frac{1}{1 - a} + \frac{U - iV}{2} - \frac{l^2(U + iV)}{2(1 - a)^2} = 0. \tag{6.23}$$

Solving (6.23) for the circulation, we obtain

$$\varkappa = \pi i\left[\frac{l^2(U + iV)}{1 - a} - (U - iV)(1 - a)\right]. \tag{6.24}$$

From Figure 6.8 it is clear that

$$1 - a = le^{-i\theta}. \tag{6.25}$$

Using (6.25) and setting

$$W = U + iV = |W|e^{i\alpha},$$

we can write (6.24) as

$$\varkappa = -2\pi|W|l\sin(\theta + \alpha). \tag{6.26}$$

Then (6.22) takes the form

$$f(z) = \frac{|W|l}{a}\left[-\frac{2\sin(\theta+\alpha)}{i}\operatorname{Ln}(le^{-i\theta} - 1 + z + \sqrt{z^2-1})\right.$$
$$+ \frac{e^{-i\alpha}}{l}(le^{-i\theta} - 1 + z + \sqrt{z^2-1})$$
$$\left. + e^{i\alpha}\frac{1}{le^{-i\theta} - 1 + z + \sqrt{z^2-1}}\right] + \text{const},$$

where we have absorbed the term $(\varkappa/2\pi i)\operatorname{Ln}(1/l)$ into the constant.

Because of the pressure forces associated with the flow, the wing-shaped contour L experiences a *lifting force*. We now calculate this force, denoted by $X + iY$ (the lift must be parallel to the xy-plane, since the flow is plane-parallel). According to Problem 6.12, the lift per unit length of wing is given by

$$X - iY = \tfrac{1}{2} i\rho \int_L [f'(z)]^2\, dz, \tag{6.27}$$

where ρ is the density of the fluid (usually air).[15] To evaluate the integral (6.27), we need only calculate the residue of $[f'(z)]^2$ at infinity (see Sec. 8). But at infinity we have the expansion

$$[f'(z)]^2 = \frac{|W|^2l^2}{4}\left[-\frac{2\sin(\theta+\alpha)}{i}\frac{1}{le^{-i\theta} - 1 + z + \sqrt{z^2-1}} + \frac{e^{-i\alpha}}{l}\right.$$
$$\left. - le^{-i\alpha}\frac{1}{(le^{i\theta} - 1 + z + \sqrt{z^2-1})^2}\right]\left(1 + \frac{z}{\sqrt{z^2-1}}\right)^2$$
$$= \frac{|W|^2l^2}{4}\left[\frac{e^{-i\alpha}}{l} + \frac{i\sin(\theta+\alpha)}{z} + \cdots\right]^2\left(2 + \frac{1}{2z^2} + \cdots\right)^2$$
$$= |W|^2e^{-2i\alpha} + 2i|W|^2le^{-i\alpha}\sin(\theta+\alpha)\frac{1}{z} + \cdots$$

It follows that

$$\operatorname*{Res}_{z=\infty}[f'(z)]^2 = -2i|W|^2le^{-i\alpha}\sin(\theta+\alpha),$$

and hence

$$X - iY = -2\pi i|W|^2l\rho e^{-i\alpha}\sin(\theta+\alpha)$$

[recall that in (6.27), the contour L is traversed in the counterclockwise direction]. Using (6.26), we find that

$$X - iY = i|W|e^{-i\alpha}\varkappa\rho = i(U - iV)\varkappa\rho,$$

or finally

$$X + iY = -i(U + iV)\varkappa\rho.$$

This is the celebrated *Kutta-Joukowski theorem*, which states that *an airfoil at rest in a uniform wind of speed* $|W| = \sqrt{U^2 + V^2}$, *with circulation* $\varkappa$ *around*

[15] Here we assume that L is rectifiable. In the integral (6.27), L can be replaced by any other closed rectifiable Jordan curve Λ such that $L \subset I(\Lambda)$.

the airfoil, undergoes a lift $|W|\varkappa\rho$ *perpendicular to the wind, where* ρ *is the air density, and the direction of the lift is obtained by rotating the velocity vector through* 90° *in the direction opposite to the circulation.*

PROBLEMS

6.1. Show that in the definitions of the normal and tangential components of the flow given on p. I.175, we can replace the requirement that L be piecewise smooth by the weaker requirement that L be rectifiable. Show that the definitions of irrotational and solenoidal flows given on p. 176 are susceptible to the same refinement. Why does this refinement represent needless sophistication?

Hint. See Lemma 2′, p. I.310 and Problem 5.4.

6.2. Verify that the quantity $\operatorname{Im} \bar{a}_1 a_2$ times a unit vector orthogonal to the xy-plane (in the same direction as the third coordinate axis in a right-handed rectangular coordinate system) is the *vector product* of the complex numbers a and b, regarded as two-dimensional vectors.[16]

6.3. Suppose a fluid element in a flow $w = u + iv$ suddenly "freezes," and afterwards moves freely with respect to the rest of the fluid. Prove that the element rotates with angular velocity

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\operatorname{curl} w.$$

Hint. Immediately after "freezing," the element undergoes translational motion, with velocity $u_0 + iv_0$, plus rotational motion about its center of mass (x_0, y_0), with angular velocity Ω. Thus the total velocity at a point (x, y) of the "frozen element" is $u + iv = [u_0 - \Omega(y - y_0)] + i[v_0 + \Omega(x - x_0)]$. But then

$$\frac{1}{2}\operatorname{curl} w = \frac{1}{2}\operatorname{curl}(u + iv) = \Omega.$$

6.4. Suppose the curl of the velocity field has the constant value a on the closed disk $\bar{K}: |z - z_0| \leqslant r$, and let L be any closed contour whose interior contains $\bar{K}$. Prove that if $r \to 0$ and $a \to \infty$ in such a way that $\pi r^2 a \to \varkappa$, then the circulation around L equals $\varkappa$, corresponding to a vortex of strength $\varkappa$ at the point z_0 (in the case of negative $\varkappa$, let $a \to -\infty$).

6.5. Let $\bar{K}$ and L be the same as in the preceding problem, but suppose the divergence of the velocity field has the constant value b on $\bar{K}$. Prove that if $r \to 0$ and $b \to \infty$ in such a way that $\pi r^2 b \to m$, then the flux through L equals m, corresponding to a source of strength m at z_0 (in the case of negative m, let $b \to -\infty$).

6.6. What are the physical dimensions of the complex potential?

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[16] See e.g., D. V. Widder, *op. cit.*, p. 63.

6.7. Analyze the flow described by each of the following complex potentials $f(z)$, i.e., plot the velocity field, equipotentials and streamlines, and look for stagnation points, sources, vortices, etc:

a) $f(z) = \ln(z^2 - a^2)$ $(a > 0)$; b) $f(z) = \ln \dfrac{z^2 - a^2}{z^2 + a^2}$ $(a > 0)$;

c) $f(z) = \ln\left(1 + \dfrac{1}{z^2}\right)$; d) $f(z) = az + \dfrac{\varkappa}{2\pi i} \ln z$ $(a > 0, \varkappa > 0)$;

e) $f(z) = \dfrac{m}{2\pi}\left(\ln z - \dfrac{1}{z}\right)$ $(m > 0)$.

Ans. b) Sources of strength 2π at the points $\pm a$, sinks of strength 2π at $\pm ia$, equipotentials $r^4 + Ca^2r^2 \cos 2\theta + a^4 = 0$ (C a real constant, $|C| > 2$) including the lines $y = \pm x$, streamlines $r^4 + Ca^2r^2 \sin 2\theta - a^4 = 0$ (C a real constant) including the coordinate axes, velocity $w = u + iv = 4a^2\bar{z}(\bar{z}^4 - a^4)$ vanishing at infinity, stagnation point at the origin, flow pattern depicted in Figure 6.9 (the pattern is the same in the other quadrants);

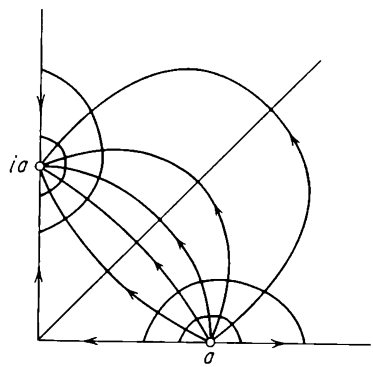

FIGURE 6.9

d) A vortex of strength $\varkappa$ at the origin, a vortex of strength $-\varkappa$ at infinity, a pole at infinity (interpreted as in Example 8, p. 187), equipotentials $ar \cos\theta = \text{const} - (\varkappa\theta/2\pi)$ including the imaginary axis, streamlines $ar \sin\theta = \text{const} - (\varkappa \ln r/2\pi)$, velocity $w = u + iv = a + (\varkappa/2\pi r) \exp[-i(\theta + \frac{1}{2}\pi)]$ equal to a at infinity, stagnation point $z = i\varkappa/2\pi a$, flow pattern depicted in Figure 6.10.

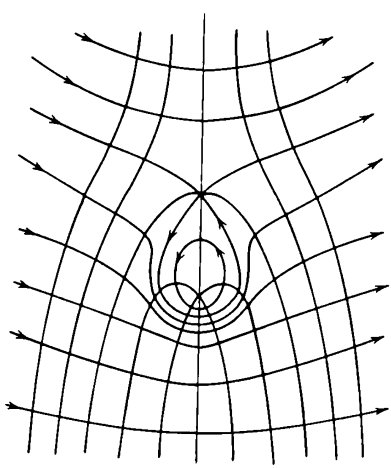
FIGURE 6.10

6.8. Find the complex potential of the flow in the upper half-plane satisfying the following requirements (w_∞ denotes the velocity at infinity):

a) $w_\infty = W$ (W real);

b) $w_\infty = 0$ with a vortex of strength $\varkappa$ at $z = a$;

c) $w_\infty = 0$ with a source of strength m at $z = a$;

d) $w_\infty = W$ with a spiral vortex of strength $m - i\varkappa$ at the point $z = a$.

Hint. The real axis must be a streamline.

Ans. a) $f(z) = Wz + \text{const}$; c) $f(z) = \dfrac{m}{2\pi} \ln[(z - a)(z - \bar{a})] + \text{const}$.

6.9. In part c of the preceding problem, there is a sink of strength $2m$ at infinity. Explain why this does not contradict the fact that the source at $z = a$ is only of strength m.

6.10. Construct a flow inside the circle $|z| = R$ with a vortex of strength $\varkappa$ at the point $z = a$ ($|a| < R$).

6.11. Under what conditions can a flow be constructed inside the circle $|z| = R$ with n sources of strengths $m_1, \ldots, m_n$ at the points $a_1, \ldots, a_n$ ($|a_k| < R$, $k = 1, \ldots, n$)? What about the case where some of the sources are on the circle itself? Find the corresponding complex potentials.

6.12. Suppose a cylindrical object, whose projection on the xy-plane is the region bounded by the contour L, is immersed in a flow described by the complex potential $f(z)$. Let X and Y be the x and y-components of the force, and M the moment about the origin, experienced by (a unit length of) the object. Prove the following formulas, known as *Blasius' theorem*[17] (ρ is the density of the fluid):

$$\begin{aligned} X - iY &= \tfrac{1}{2}i\rho \int_L [f'(z)]^2\, dz, \\ M &= -\tfrac{1}{2}\rho \operatorname{Re} \int_L z[f'(z)]^2\, dz. \end{aligned} \tag{6.28}$$

Hint. If $p = p(x, y)$ is the pressure in the flow $w = u + iv$, then, according to *Bernoulli's theorem*,[18] the quantity

$$\frac{p}{\rho} + \frac{1}{2}|w|^2$$

has the same value at every point of a given streamline. The contributions to X, Y and M due to the pressure forces acting on an element $dz = dx + i\,dy$ of the contour L are

$$dX = -p\,dy, \qquad dY = p\,dx, \qquad dM = p(x\,dx + y\,dy)$$

(draw a figure), so that

$$d(X - iY) = -ip\,d\bar{z}, \qquad dM = \operatorname{Re}(pz\,d\bar{z}).$$

Using Bernoulli's theorem, integrate these expression around L, recalling that L must be a streamline.

6.13. Derive the complex potential $f(z)$ for flow past an elliptic cylinder with semiaxes a and b, where the flow has velocity $U + iV$ at infinity and circulation $\varkappa$.

Hint. Use (6.18) after verifying that

$$\omega = F(z) = \frac{1}{a + b}(z + \sqrt{z^2 - c^2}) \qquad (c^2 = a^2 - b^2)$$

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[17] Also associated with the name of S. A. Chaplygin.

[18] See e.g., L. M. Milne-Thomson, *op. cit.*, p. 8.

(choose the positive square root) maps the exterior of the ellipse with semiaxes a and b onto the exterior of the circle $|\omega| = 1$.

Ans.

$$f(z) = \frac{\varkappa}{2\pi i}\operatorname{Ln}(z + \sqrt{z^2 - c^2}) + \frac{U}{a - b}[az - b\sqrt{z^2 - c^2}] + \frac{iV}{a - b}[bz - a\sqrt{z^2 - c^2}] + \text{const.} \tag{6.29}$$

6.14. Introducing elliptic coordinates ξ and η, related to the rectangular coordinates x and y by the formula

$$x = c\cosh\xi\cos\eta, \qquad y = c\sinh\xi\sin\eta,$$

or more concisely,

$$z = c\cosh\zeta \qquad (\zeta = \xi + i\eta),$$

prove that in the absence of circulation, (6.29) takes the form

$$f(z) = |W|(a + b)\cosh(\zeta - \xi_0 - i\alpha) + \text{const}, \tag{6.30}$$

where $W = U + iV = |W|e^{i\alpha}$, and ξ_0 is the parameter corresponding to the ellipse with semiaxes a and b, i.e.,

$$a = c\cosh\xi_0, \qquad b = c\sinh\eta_0.$$

Verify that this ellipse is indeed a streamline, and find the stagnation points of the flow. Why would one expect the ellipse to be subject to a moment tending to turn it broadside into the flow?

6.15. Calculate the moment M acting on an elliptic cylinder with semiaxes a and b immersed in a flow with no circulation and velocity $|W|e^{i\alpha}$ at infinity. Verify that the cylinder undergoes no lift.

Hint. Substitute (6.29) or (6.30) into (6.28), and evaluate the integral by residue calculus, as in the derivation of the Kutta-Joukowski theorem.

Ans. $M = -\frac{1}{2}\pi\rho c^2|W|^2\sin 2\alpha$.

6.16. Particularize the theory of the preceding three problems to the case of flow past a strip of width $2a$, and show that the velocity is infinite at the edges of the strip. Show that by introducing circulation, the velocity at one edge of the strip can be made finite. Calculate the lift associated with this circulation.

7

SUBHARMONIC FUNCTIONS

32. The Key Lemma. The Converse of Theorem 5.6

We now generalize the concept of a harmonic function in a way that has far-reaching implications. A real function $h(z) = h(x, y)$ is said to be *subharmonic* on a domain G if it has the following properties:[1]

1. $h(x, y)$ is defined and continuous at every point of G except possibly at a finite number of points, or at the points of a sequence $\{(a_n, b_n)\}$ which has no limit points in G, while every exceptional point (a_n, b_n) satisfies the relation

$$\lim_{(x,y)\to(a_n,b_n)} h(x, y) = -\infty, \tag{7.1}$$

which is used to define

$$h(a_n, b_n) = -\infty; \tag{7.2}$$

2. The integral

$$\frac{1}{2\pi}\int_0^{2\pi} h(z_0 + \rho e^{i\varphi})\, d\varphi$$

exists for every point $z_0 \in G$ and all ρ less than $\Delta(z_0)$, the distance between z_0 and the boundary of G;

3. The inequality

$$h(z_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + \rho e^{i\varphi})\, d\varphi \tag{7.3}$$

holds for every point $z_0 \in G$ and all *sufficiently small* $\rho > 0$.

[1] This is not the most general (or the least redundant) definition of a subharmonic function, but is sufficient for our purposes.

Example 1. For every function $u(z) = u(x, y)$ harmonic on a domain G the inequality (7.3) becomes an equality

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\varphi})\, d\varphi$$

(see Theorem 5.6). Therefore, according to our definition, every harmonic function is subharmonic.

Example 2. If $f(z)$ is analytic on a domain G, then $|f(z)|$ is subharmonic on G. In fact,

$$|f(z_0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\varphi})|\, d\varphi \tag{7.4}$$

for every $z_0 \in G$ and all sufficiently small ρ (cf. Theorem I.14.2). Moreover, it is not hard to show that equality is possible in (7.4) if and only if $f(z) \equiv \text{const}$ (see Problem 7.4). Therefore $|f(z)|$ is subharmonic but not harmonic on G, unless $f(z) \equiv \text{const}$.

Example 3. If $f(z) \not\equiv 0$ is analytic on a domain G, then $\ln |f(z)|$ is subharmonic on G. In fact, $\ln |f(z)|$ is harmonic everywhere in G except at the zeros of $f(z)$, where it has logarithmic poles and takes the value $-\infty$. If z_0 is not a zero of $f(z)$, then

$$\ln |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln |f(z_0 + \rho e^{i\varphi})|\, d\varphi$$

for all sufficiently small ρ, while if z_0 is a zero of $f(z)$, then

$$\ln |f(z_0)| = -\infty,$$

and hence

$$\ln |f(z_0)| < \frac{1}{2\pi}\int_0^{2\pi} \ln |f(z_0 + \rho e^{i\varphi})|\, d\varphi.$$

LEMMA 1. *Let $h(z)$ be a real function defined on a bounded domain G, and let*

$$\sup_{z \in G} h(z) = M \tag{7.5}$$

(M may be infinite). Then there is at least one point $\zeta \in \overline{G}$ such that

$$\sup_{z \in \mathcal{N}(\zeta) \cap G} h(z) = M \tag{7.6}$$

for every neighborhood $\mathcal{N}(\zeta)$.

Proof. Suppose the lemma is false. Then every point $\zeta \in \overline{G}$ has a neighborhood $\mathcal{N}(z)$ such that

$$\sup_{z \in \mathcal{N}(\zeta) \cap G} h(z) = M(\zeta) < M.$$

Thus, according to the Heine-Borel theorem (Theorem I.3.14), there exists a finite number of points $z_1, \ldots, z_n$ in $\bar{G}$ such that

$$\bar{G} \subset \mathcal{N}(z_1) \cup \cdots \cup \mathcal{N}(z_n).$$

Therefore, if

$$M_0 = \max \{M(z_1), \ldots, M(z_n)\},$$

then

$$\sup_{z \in \bar{G}} h(z) = M_0 < M,$$

contrary to (7.5). This contradiction proves the lemma.

The following result will serve as a basic tool in our investigation of subharmonic functions:

LEMMA 2 (*Key lemma*). *Let $h(z)$ be subharmonic on a bounded domain G with boundary Γ, and suppose that*

$$\overline{\lim_{z \to \zeta}} h(z) \leqslant 0 \tag{7.7}$$

for all $\zeta \in \Gamma$. Then

$$h(z) \leqslant 0$$

for all $z \in G$. Moreover, if $h(z_0) = 0$ for any point $z_0 \in G$, then

$$h(z) \equiv 0.$$

Proof. Let

$$\sup_{z \in G} h(z) = M,$$

and let E be the set of all points $\zeta \in \bar{G}$ such that (7.6) holds. According to Lemma 1, E is nonempty. Suppose first that no interior point of G belongs to E. Then E contains at least one point $\zeta_0 \in \Gamma$. Because of (7.7), given any $\varepsilon > 0$, there is a neighborhood $\mathcal{N}(\zeta_0)$ in which $h(z) < \varepsilon$. Therefore on the one hand,

$$\sup_{z \in \mathcal{N}(\zeta_0) \cap G} h(z) \leqslant \varepsilon,$$

while on the other hand,

$$\sup_{z \in \mathcal{N}(\zeta_0) \cap G} h(z) = M \tag{7.8}$$

(since $\zeta_0 \in E$). It follows that $M \leqslant \varepsilon$ and hence $M \leqslant 0$, where $h(z) < M$ at points of G. This proves the lemma for the case where E contains no points of G [note that this case does not apply if $h(z_0) = 0$ for $z_0 \in G$].

Next suppose E contains points of G, and let $E_G = E \cap G$. Then since the subharmonic function $h(z)$ is (wide-sense) continuous on G, $h(z)$ must equal M at all points of E_G. In fact, any point $z_0 \in G$ at which $h(z_0) < M$ has a neighborhood $\mathcal{N}(z_0)$ in which $h(z) < M - \delta$ for some $\delta > 0$, and hence (7.6) cannot hold at such a point. In

particular, it follows that $M < +\infty$, being one of the values of a subharmonic function. On the other hand $M > -\infty$, since a subharmonic function takes the value $-\infty$ only at certain exceptional points of G.

We now show that E_G is an open set. Given $z_0 \in E_G$, let $\rho_0 > 0$ be such that the inequality

$$M = h(z_0) \leqslant \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + \rho e^{i\varphi})\, d\varphi \tag{7.9}$$

holds for all $\rho < \rho_0$ [the fact that $h(z)$ is subharmonic implies the existence of such a ρ_0]. Then the disk $|z - z_0| < \rho_0$ contains no points at which $h(z) < M$. In fact, if z_1 were such a point, z_1 would have a neighborhood in which $h(z) < M - \delta$ for some $\delta > 0$, and hence for $\rho = |z_1 - z_0|$ the integral in (7.9) would be less than M, which is impossible. Therefore every point $z_0 \in E_G$ has a neighborhood $\mathcal{N}(z_0)$ in which $h(z) = M$, i.e., $\mathcal{N}(z_0) \subset E_G$ and E_G is open. Moreover, suppose $z' \in G$ is a limit point of E_G. Then, since $h(z) = M$ at every point of E_G and since $h(z)$ is continuous on G, $h(z')$ must equal M and hence $z' \in E_G$.

Thus E_G is a nonempty open subset of G, with the property that every limit point of E_G belonging to G also belongs to E_G. Therefore, according to Theorem I.4.16, $E_G = G$ and hence $h(z) \equiv M$. But then

$$\overline{\lim_{z \to \zeta}}\, h(z) = \lim_{z \to \zeta} h(z) = M$$

for every boundary point $\zeta \in \Gamma$, where $M \leqslant 0$ because of (7.7). Therefore, whether or not E contains points of G, we have

$$\sup_{z \in G} h(z) = M \leqslant 0.$$

Moreover, if $h(z_0) = 0$ at any point $z_0 \in G$, then $M = 0$ and hence $h(z) \equiv 0$. This completes the proof of the key lemma.

It is now an easy matter to prove the converse of Theorem 5.6:

THEOREM 7.1. *Let $u(x, y)$ be a continuous real function on a domain G, and suppose $u(x, y)$ satisfies the condition*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + \rho \cos \varphi, y_0 + \rho \sin \varphi)\, d\varphi$$

for every point $z_0 = (x_0, y_0) \in G$ and all sufficiently small ρ. Then $u(x, y)$ is harmonic on G.

Proof. Clearly we need only prove the result in some neighborhood of an arbitrary point $z_0 \in G$. Let the circle $\gamma_\rho: |z - z_0| = \rho$ be such that $\overline{I(\gamma_\rho)} \subset G$, and use Theorem 5.8 to construct the (unique) function

$u^*(x, y)$ which is harmonic on $I(\gamma_\rho)$ and coincides with $u(x, y)$ on γ_ρ. Then the difference $u^*(x, y) - u(x, y)$ is continuous on $\overline{I(\gamma_\rho)}$ and satisfies the relation

$$u^*(x_1, y_1) - u(x_1, y_1) = \frac{1}{2\pi}\int_0^{2\pi} [u^*(x_1 + \rho_1 \cos\varphi, y_1 + \rho_1 \sin\varphi) - u(x_1 + \rho_1 \cos\varphi, y_1 + \rho_1 \sin\varphi)]\, d\varphi$$

for every $z_1 = (x_1, y_1) \in I(\gamma_\rho)$ and all sufficiently small ρ_1. Therefore $u^*(x, y) - u(x, y)$ is subharmonic on $I(\gamma_\rho)$. But, by construction, $u^*(x, y) - u(x, y)$ vanishes at every point of the circle γ_ρ, and hence also at its center z_0, since

$$u^*(x_0, y_0) - u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} [u^*(x_0 + \rho \cos\varphi, y_0 + \rho \sin\varphi) - u(x_0 + \rho \cos\varphi, y_0 + \rho \sin\varphi)]\, d\varphi = 0.$$

Therefore, by the key lemma,

$$u^*(x, y) - u(x, y) = 0$$

for all $(x, y) \in I(\gamma_\rho)$, i.e., $u(x, y)$ is harmonic in a neighborhood of (x_0, y_0). Since (x_0, y_0) is an arbitrary point of G, the proof is now complete.

33. The Generalized Maximum Modulus Principle and Its Applications

Another important consequence of the key lemma is

THEOREM 7.2 (*Generalized maximum modulus principle*). *Let $h(z) = h(x, y)$ be subharmonic on a bounded domain G with boundary Γ, and let $u(z) = u(x, y)$ be harmonic on G. Suppose that*

$$\overline{\lim_{z \to \zeta}} [h(z) - u(z)] \leqslant 0 \tag{7.10}$$

for all $\zeta \in \Gamma$ except possibly for a finite number of points $\zeta_1, \ldots, \zeta_n$ where

$$\overline{\lim_{z \to \zeta_k}} [h(z) - u(z)] < +\infty. \tag{7.11}$$

Then $u(z)$ is a "harmonic majorant" of $h(z)$ on G, i.e.,

$$h(z) \leqslant u(z)$$

for all $z \in G$. Moreover, if $h(z_0) = u(z_0)$ for any point $z_0 \in G$, then

$$h(z) \equiv u(z).$$

Proof. Roughly speaking, the theorem asserts that a harmonic function $u(z)$ which *majorizes* a subharmonic function $h(z)$ [i.e., is

$\geqslant h(z)]$ at all but a finite number of points of the boundary of a domain G also majorizes $h(z)$ at all points of G itself. In allowing exceptional points where the condition (7.10) does not hold, we anticipate subsequent applications of Theorem 7.2. The fact that the exceptional points must satisfy the condition (7.11) in order for the theorem to be true is shown by the example of the Poisson kernel

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi_0)},$$

which is harmonic and hence subharmonic on the disk $|z| < \rho$ (see Example 4, p. 146), and satisfies the condition

$$\lim_{(r,\theta) \to (\rho,\varphi)} P(r, \theta) = 0$$

at every point of the circle $\gamma_\rho: |z| = \rho$ except the one point $z = \rho e^{i\varphi_0}$. Therefore $P(r, \theta)$ is majorized by the harmonic function $u(r, \theta) \equiv 0$ at every point of γ_ρ except at $z = \rho$. However

$$P(r, \theta) > 0 = u(r, \theta)$$

for all $z \in I(\gamma_\rho)$. The explanation is to be found in the fact that $P(r. \theta)$ takes arbitrarily large values near the point $z = \rho$, and therefore neither the function which is identically zero nor any other bounded function can majorize $P(r, 0)$ on $I(\gamma_\rho)$.

Having made these preliminary observations, we turn to the proof itself. Consider the functions

$$v_k(z) = \ln |z - \zeta_k| \qquad (k = 1, \ldots, n).$$

Obviously $v_k(z)$ is harmonic on G and takes the value $-\infty$ at $z = \zeta_k$. If

$$R = \operatorname{diam}(\bar{G})$$

(see Problem I.3.27), then

$$v_k(z) \leqslant \ln R \qquad (z \in \bar{G})$$

for every function $v_k(z)$. Therefore the functions

$$u_k(z) = v_k(z) - \ln R \qquad (k = 1, \ldots, n)$$

cannot take positive values on $\bar{G}$, but otherwise have the same properties as the functions $v_k(z)$.

Now let ε be a positive parameter, which will subsequently be made to approach zero, and consider the function

$$d_\varepsilon(z) = h(z) - u(z) + \varepsilon \sum_{j=1}^{n} u_j(z).$$

Given any point $z_0 \in G$, $h(z_0)$ does not exceed the average of the values of $h(z)$ on any sufficiently small circle with center z_0, while $u(z_0)$ and

$u_k(z_0)$ coincide with the corresponding "circle averages." Therefore $d_\varepsilon(z_0)$ does not exceed the average of $d_\varepsilon(z)$ on any sufficiently small circle with center z_0, and hence is subharmonic on G. Moreover, $d_\varepsilon(z)$ satisfies the relation

$$\overline{\lim_{z \to \zeta}}\, d_\varepsilon(z) = \overline{\lim_{z \to \zeta}} \left[h(z) - u(z) + \varepsilon \sum_{j=1}^{n} u_j(z) \right] \leqslant 0 \qquad (7.12)$$

at every boundary point $\zeta \neq \zeta_k$, and the relation

$$\overline{\lim_{z \to \zeta_k}}\, d_\varepsilon(z) = -\infty \leqslant 0 \qquad (k = 1, \ldots, n)$$

at every boundary point $\zeta = \zeta_k$ [since the functions $h(z) - u(z)$ and $u_j(z)$, $j \neq k$ are bounded from above in a neighborhood of ζ_k, while $u_k(z) \to -\infty$ as $z \to \zeta_k$]. Therefore the key lemma applies to the function $d_\varepsilon(z)$, and hence

$$d_\varepsilon(z) \leqslant 0 \qquad (z \in G),$$

or

$$h(z) \leqslant u(z) - \varepsilon \sum_{j=1}^{n} u_j(z).$$

Letting ε approach zero, we obtain

$$h(z) \leqslant u(z) \qquad (z \in G),$$

so that $u(z)$ is a harmonic majorant of $h(z)$ on G, as asserted.

Finally, we observe that

$$\overline{\lim_{z \to \zeta}} \left[h(z) - u(z) \right] \leqslant 0$$

for every $\zeta \in \Gamma$, including the exceptional points $\zeta_1, \ldots, \zeta_n$. Therefore, by the key lemma, if

$$h(z_0) - u(z_0) = 0$$

for any point $z_0 \in G$, then $h(z) - u(z) = 0$ for all $z \in G$, i.e., $h(z) \equiv u(z)$ and the theorem is proved.

COROLLARY. *Let $f(z)$ be analytic on a bounded domain G with boundary Γ, and suppose that*

$$\overline{\lim_{z \to \zeta}} |f(z)| \leqslant M < \infty$$

for all $\zeta \in \Gamma$ except for a finite number of points $\zeta_1, \ldots, \zeta_n$ where

$$\overline{\lim_{z \to \zeta_k}} |f(z)| < \infty.$$

Then

$$|f(z)| \leqslant M$$

for all $z \in G$. *Moreover, if* $|f(z_0)| = M$ *for any point* $z_0 \in G$, *then*

$$f(z) \equiv M.$$

Proof. Choose $h(z) = |f(z)|$ and $u(z) \equiv M$.

As our first application of the generalized maximum modulus principle, we now prove

THEOREM 7.3 (*Hadamard's three-circles theorem*). *Let* $f(z)$ *be analytic on an annulus* $D: r_1 < |z| < r_2$, *and suppose that*

$$\overline{\lim_{\substack{z \to \zeta \\ |\zeta| = r_1}}} |f(z)| \leqslant \ln M_1, \qquad \overline{\lim_{\substack{z \to \zeta \\ |\zeta| = r_2}}} |f(z)| \leqslant \ln M_2.$$

Then

$$\ln M(\rho) \leqslant \ln M(\rho_1) \frac{\ln \frac{\rho}{\rho_2}}{\ln \frac{\rho_1}{\rho_2}} + \ln M(\rho_2) \frac{\ln \frac{\rho}{\rho_1}}{\ln \frac{\rho_2}{\rho_1}}, \tag{7.13}$$

where $r_1 < \rho_1 < \rho < \rho_2 < r_2$ *and*

$$M(\rho) = \max_{|z| = \rho} |f(z)|.$$

Moreover, if equality holds in (7.13), *then*

$$f(z) = Az^{(\ln M_2 - \ln M_1)/(\ln r_2 - \ln r_1)}, \tag{7.14}$$

where

$$A = \exp\left(iC + \frac{\ln M_1 \ln r_2 - \ln M_2 \ln r_1}{\ln r_2 - \ln r_1}\right). \tag{7.15}$$

Proof. The function

$$u(z) = \ln M_1 \frac{\ln \frac{|z|}{r_2}}{\ln \frac{r_1}{r_2}} + \ln M_2 \frac{\ln \frac{|z|}{r_1}}{\ln \frac{r_2}{r_1}} \qquad (r_1 < |z| < r_2),$$

which is obviously harmonic on D and continuous on $\bar{D}$, reduces to $\ln M_1$ on the circle $|z| = r_1$ and to $\ln M_2$ on the circle $|z| = r_2$. Therefore, according to Theorem 7.2, $u(z)$ is a harmonic majorant of $\ln |f(z)|$ on D, i.e.,

$$\ln |f(z)| \leqslant \ln M_1 \frac{\ln \frac{|z|}{r_2}}{\ln \frac{r_1}{r_2}} + \ln M_2 \frac{\ln \frac{|z|}{r_1}}{\ln \frac{r_2}{r_1}} \qquad (r_1 < |z| < r_2). \tag{7.16}$$

The inequality (7.13) is an immediate consequence of (7.16) and the definition of $M(\rho)$. Moreover, equality can hold in (7.16) only if

$\ln |f(z_0)| = u(z_0)$ for some point $z_0 \in D$, in which case $\ln |f(z)| \equiv u(z)$. But then

$$v(z) = \ln M_1 \frac{\operatorname{Arg} \frac{z}{r_2}}{\ln \frac{r_1}{r_2}} + \ln M_2 \frac{\operatorname{Arg} \frac{z}{r_1}}{\ln \frac{r_2}{r_1}},$$

which is obviously the conjugate harmonic function of $u(z)$, can only differ by a real constant C from $\operatorname{Arg} f(z)$, the conjugate harmonic function of $\ln |f(z)|$. In other words,

$$\begin{aligned} \operatorname{Ln} f(z) = \ln |f(z)| + i \operatorname{Arg} f(z) &= iC + u(z) + iv(z) \\ &= iC + \ln M_1 \frac{\ln \frac{z}{r_2}}{\ln \frac{r_1}{r_2}} + \ln M_2 \frac{\ln \frac{z}{r_1}}{\ln \frac{r_2}{r_1}}, \end{aligned}$$

which is equivalent to (7.14) and (7.15).

Remark. Theorem 7.3 has a simple geometric interpretation, which becomes apparent when we consider $\eta = \ln M(\rho)$ as a function of $\xi = \ln \rho$:

$$\eta = \varphi(\xi). \tag{7.17}$$

Setting $\xi_1 = \ln \rho_1$, $\xi_2 = \ln \rho_2$ and $\eta_1 = \ln M(\rho_1)$, $\eta_2 = \ln M(\rho_2)$, we can write (7.13) in the form

$$\eta \leqslant \eta_1 \frac{\xi_2 - \xi}{\xi_2 - \xi_1} + \eta_2 \frac{\xi - \xi_1}{\xi_2 - \xi_1}, \tag{7.18}$$

where $\xi_1 < \xi < \xi_2$. But in rectangular coordinates ξ and η,

$$\eta = \eta_1 \frac{\xi_2 - \xi}{\xi_2 - \xi_1} + \eta_2 \frac{\xi - \xi_1}{\xi_2 - \xi_1} \tag{7.19}$$

is the equation of the straight line joining the points (ξ_1, η_1) and (ξ_2, η_2). Comparing (7.18) and (7.19), we see that no arc joining two points of the curve with equation (7.17) can lie higher than the chord with the same end points. Both the curve with this property and the corresponding function $\varphi(\xi)$ are said to be *convex*. Thus Hadamard's three-circles theorem has the following concise formulation: *The logarithm of the maximum modulus of a function analytic on an annulus $D: r_1 < |z| < r_2$ is a convex function of the logarithm of the radius of the circle on which the maximum is taken.*

Another important application of the generalized maximum modulus principle is given by

THEOREM 7.4 (*Two-constants theorem for a disk*). *Let $f(z)$ be analytic on the disk $K: |z| < R$, and let ζ_1 and ζ_2 be two points of the circle $\Gamma: |z| = R$. Let σ be the arc $\overset{\frown}{\zeta_1\zeta_2}$ and Σ the arc $\overset{\frown}{\zeta_2\zeta_1}$, both traversed*

in the positive direction and both without their end points,[2] *and suppose that*

$$\overline{\lim_{\substack{z \to \zeta \\ \zeta \in \sigma}}} \ln |f(z)| \leqslant \ln m, \qquad \overline{\lim_{\substack{z \to \zeta \\ \zeta \in \Sigma}}} \ln |f(z)| \leqslant \ln M, \tag{7.20}$$

while

$$\overline{\lim_{z \to \zeta_k}} \ln |f(z)| < +\infty \qquad (k = 1, 2).$$

Moreover, let 2α *and* 2β *be the angles subtended at the point* $z = 0$ *by the arcs* σ *and* Σ, *respectively* ($\alpha > 0, \beta > 0, \alpha + \beta = \pi$), *and let* $U(z)$ *be the single-valued harmonic branch of the function*

$$\operatorname{Arg} \frac{\zeta_1 - z}{\zeta_2 - z}$$

such that $U(0) = 2\beta$. *Then*

$$|f(z)| \leqslant M^{u(z)} m^{1-u(z)} \tag{7.21}$$

for all $z \in K$, *where*

$$u(z) = \frac{1}{\pi} [U(z) - \beta],$$

and in particular,

$$|f(0)| \leqslant M^{\beta/\pi} m^{\alpha/\pi}. \tag{7.22}$$

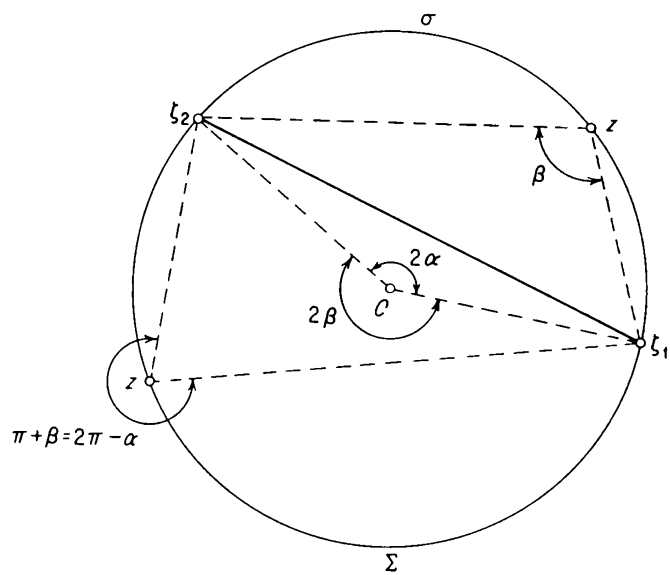

FIGURE 7.1

[2] An arc without its end points will be called an *open arc* (cf. p. I.370).

Proof. As z varies over the disk K, starting from the center $z = 0$, the function $U(z)$ varies in the interval $(0, 2\pi)$, starting from the value $U(0) = 2\beta$. It is clear from the geometric meaning of $U(z)$ that

$$U(z) = \begin{cases} \beta & \text{if } z \in \sigma, \\ \pi + \beta & \text{if } z \in \Sigma \end{cases}$$

(see Figure 7.1), and hence

$$u(z) = \frac{1}{\pi}[U(z) - \beta] = \begin{cases} 0 & \text{if } z \in \sigma, \\ 1 & \text{if } z \in \Sigma. \end{cases}$$

Moreover, both $U(z)$ and $u(z)$ are bounded in a neighborhood of ζ_1 or ζ_2. Applying Theorem 7.2, we see that the harmonic function $u(z)$ is majorized by 1 on Γ and hence on K. Similarly, the harmonic function $-u(z)$ is majorized by 0 on Γ and hence on K. Thus

$$0 < u(z) < 1 \tag{7.23}$$

for all $z \in K$, and in particular, $u(0) = \beta/\pi$.[3]

Now consider the function

$$\ln m + (\ln M - \ln m)u(z),$$

which reduces to $\ln m$ on σ and to $\ln M$ on Σ, while remaining bounded in a neighborhood of ζ_1 or ζ_2. According to (7.20), this function is a harmonic majorant of $\ln |f(z)|$ on Γ and hence on K, i.e.,

$$\ln |f(z)| \leqslant \ln m + (\ln M - \ln m)u(z)$$

or

$$\ln |f(z)| \leqslant \ln M \cdot u(z) + \ln m[1 - u(z)]$$

for all $z \in K$, which is just another way of writing (7.21).

Remark. Suppose that $m < M$, and that instead of fully exploiting the available information about the boundary values of $f(z)$, we replace (7.20) by the inequality

$$\overline{\lim_{\substack{z \to \zeta \\ \zeta \in \sigma \cup \Sigma}}} |f(z)| \leqslant M.$$

Then, according to the corollary to Theorem 7.2,

$$|f(z)| \leqslant M$$

---

[3] In writing (7.23) instead of $0 \leqslant u(z) \leqslant 1$, we use the fact that

$$\max_{|z| \leqslant r < 1} \pm u(z)$$

is a strictly increasing function of r, since $u(z) \not\equiv \text{const}$ is harmonic (see Theorem 5.7, Corollary 1).

for all $z \in K$. However, (7.21) is a much better estimate of $|f(z)|$, since

$$\frac{M^{u(z)}m^{1-u(z)}}{M} = \left(\frac{m}{M}\right)^{1-u(z)} < 1$$

[recall (7.23)].

Example. As an application of Theorem 7.4, let $\{f_n(z)\}$ be a sequence of functions which are analytic and uniformly bounded on the disk $K: |z| < R$, i.e.,

$$|f_n(z)| \leqslant M$$

for all n and $z \in K$, and suppose that

$$\overline{\lim_{\substack{z \to \zeta \\ \zeta \in \sigma}}} |f_n(z)| \leqslant \varepsilon_n,$$

where σ is some fixed open arc of the circle $\Gamma: |z| = R$ and $\{\varepsilon_n\}$ is a sequence of positive numbers converging to zero ($\varepsilon_n \leqslant M$). Then $\{f_n(z)\}$ converges uniformly to zero on every compact subset of K. To see this, we first note that

$$\overline{\lim_{\substack{z \to \zeta \\ \zeta \in \sigma}}} \ln |f_n(z)| \leqslant \ln \varepsilon_n, \qquad \overline{\lim_{\substack{z \to \zeta \\ \zeta \in \Sigma}}} \ln |f_n(z)| \leqslant \ln M,$$

where Σ is the "complementary open arc" $\Gamma - \sigma - \{\zeta_1\} - \{\zeta_2\}$, where ζ_1 and ζ_2 denote the end points of σ. It follows from Theorem 7.4 that

$$|f_n(z)| \leqslant M^{u(z)}\varepsilon_n^{1-u(z)} \leqslant \mu\varepsilon_n^{1-u(z)},$$

where $\mu = \max(M, 1)$.[4] On every closed disk $|z| \leqslant r < R$, the harmonic function $u(z)$ has a maximum $\tau(r) < 1$ (cf. footnote 3, p. 212). Therefore

$$|f_n(z)| \leqslant \mu\varepsilon_n^{1-\tau(r)}$$

if $|z| \leqslant r$, and hence $\{f_n(z)\}$ converges uniformly to zero, as asserted (every compact subset of K is contained in some disk $|z| \leqslant r < R$).

In particular, let

$$f_n(z) = f(z) \qquad (n = 1, 2, \ldots),$$

so that $f(z)$ is a bounded analytic function on K, satisfying the condition

$$\overline{\lim_{\substack{z \to \zeta \\ \zeta \in \sigma}}} |f(z)| \leqslant \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. In other words,

$$\lim_{\substack{z \to \zeta \\ \zeta \in \sigma}} f(z) = 0$$

on some arc $\sigma \subset \Gamma$. It follows that

$$f(z) = \lim_{n \to \infty} f_n(z) \equiv 0.$$

---

[4] Note that, since $0 < u(z) < 1$, $M^{u(z)} < M$ if $M > 1$, while $M^{u(z)} \leqslant 1$ if $M \leqslant 1$.

Thus a bounded analytic function on $I(\Gamma)$, with zero boundary values on some arc $\sigma \subset \Gamma$, must be identically zero on $I(\Gamma)$. This is a very special case of the Luzin-Privalov boundary uniqueness theorem cited on p. I.371 (the boundedness requirement actually turns out to be unnecessary).

*34. The Phragmén-Lindelöf Theorems

According to the corollary to Theorem 7.2, if $f(z)$ is analytic on a domain G with boundary Γ, and if

$$\overline{\lim_{\substack{z \to \zeta \\ \zeta \in \Gamma}}} |f(z)| \leqslant C < \infty$$

except at one point ζ_0 where

$$\overline{\lim_{z \to \zeta_0}} |f(z)| < \infty, \tag{7.24}$$

then $|f(z)| \leqslant C$ for all $z \in G$, and $|f(z_0)| = C$ at a point $z_0 \in G$ implies $f(z) \equiv C$. However, if we have information about the character of Γ near the exceptional point ζ_0, the condition (7.24) can often be relaxed. For example, suppose that at the point ζ_0, Γ forms an angle of $\alpha\pi$ radians $(0 < \alpha \leqslant 2)$.[5] Then we can get rid of the condition (7.24), provided that $|f(z)|$ does not grow too rapidly as $z \to \zeta_0$. For simplicity, we confine ourselves to the case where $\zeta_0 = \infty$ and G is the interior of an angle or a strip.[6]

THEOREM 7.5 (*Phragmén-Lindelöf theorem for an angle*). *Let G be the interior of an angle of $\alpha\pi$ radians $(0 < \alpha \leqslant 2)$, with boundary Γ, and let $f(z)$ be analytic on G. Suppose $f(z)$ satisfies the following conditions*:

1. *For every finite point $\zeta \in \Gamma$,*

$$\overline{\lim_{z \to \zeta}} |f(z)| \leqslant C < \infty; \tag{7.25}$$

2.
$$\lim_{r \to \infty} \frac{\ln \ln M(r)}{\ln r} < \frac{1}{\alpha}, \tag{7.26}$$

where

$$M(r) = \sup_{\substack{|z| = r \\ z \in G}} |f(z)|. \tag{7.27}$$

Then

$$|f(z)| \leqslant C \tag{7.28}$$

for all $z \in G$. Moreover, if $|f(z_0)| = C$ for some point $z_0 \in G$, then

$$f(z) \equiv C.$$

---

[5] This presupposes that Γ has left and right-hand tangents at ζ_0 (cf. p. I.126).

[6] On the Riemann sphere, every angle has a vertex at infinity, and a strip is the interior of an angle of 0 radians (see the remark on p. I.90).

Proof. It is easy to see that (7.26) is equivalent to the following restriction on the growth of $|f(z)|$ as $z \to \infty$: Given any ρ_1 less than $1/\alpha$ but sufficiently close to $1/\alpha$, there exists a sequence $\{r_n\}$, where $r_n < r_{n+1}$ $(n = 1, 2, \ldots)$ and $r_n \to \infty$ as $n \to \infty$, such that

$$|f(z)| < \exp(|z|^{\rho_1}) \quad \text{if} \quad |z| = r_n \quad (n = 1, 2, \ldots). \tag{7.29}$$

There is no loss of generality in assuming that the finite vertex of G lies at the origin and that the bisector of the angle coincides with the positive real axis, since the general case can be reduced to this one by making a preliminary entire linear transformation (consisting of a shift and a rotation), without affecting the hypotheses of the theorem.

Now consider the function

$$F_\varepsilon(z) = f(z) \exp(-\varepsilon z^{\rho_2}), \tag{7.30}$$

where $\varepsilon > 0$, $\rho_1 < \rho_2 < 1/\alpha$ and ρ_1 is the same as (7.29). Here z^{ρ_2} denotes the branch of the multiple-valued function $z^{\rho_2} = \exp(\rho_2 \operatorname{Ln} z)$ which is single-valued and analytic on G, and takes positive values for positive real z, i.e., $z^{\rho_2} = \exp(\rho_2 \ln z)$. Let G_n be the sector $|\arg z| < \alpha\pi/2$, $|z| < r_n$, bounded by two line segments and an open circular arc λ_n. According to (7.25), for every point ζ belonging to these line segments,

$$\overline{\lim_{z \to \zeta}} |F_\varepsilon(z)| \leqslant C \lim_{z \to \zeta} |\exp(-\varepsilon z^{\rho_2})| = C \lim_{z \to \zeta} \exp(-\varepsilon r^{\rho_2} \cos \rho_2\theta) \leqslant C,$$

where $z = re^{i\theta}$ (in the domain G, we have

$$|\rho_2\theta| < \frac{1}{\alpha}\frac{\alpha\pi}{2} = \frac{\pi}{2},$$

and hence $\cos \rho_2\theta > 0$). Moreover, according to (7.29), on the arc λ_n,

$$|F_\varepsilon(z)| < \exp(r_n^{\rho_1} - \varepsilon r_n^{\rho_2} \cos \rho_2\theta) \exp\left(r_n^{\rho_1} - \varepsilon r_n^{\rho_2} \cos \frac{\rho_2\alpha\pi}{2}\right),$$

where the expression on the right approaches zero as $n \to \infty$, since $\cos(\rho_2\alpha\pi/2) > 0$ and $\rho_1 < \rho_2 < 1/\alpha$. Choosing n so large that $|F_\varepsilon(z)| \leqslant C$ on λ_n, we have

$$\overline{\lim_{z \to \zeta}} |F_\varepsilon(z)| \leqslant C$$

for all ζ on the boundary of G_n. Therefore, by Theorem 7.2,

$$|F_\varepsilon(z)| \leqslant C$$

for all $z \in G_n$. Substituting from (7.30) and letting the parameter ε approach zero, we find that

$$\lim_{\varepsilon \to 0} |f(z) \exp(-\varepsilon z^{\rho_2})| = |f(z)| \leqslant C$$

for all $z \in G_n$ if n is sufficiently large. This proves (7.28), since n can always be chosen large enough to make G_n contain any preassigned

point of G. Finally, we note that if $|f(z_0)| = C$ for any point $z_0 \in G$, then $f(z) \equiv C$ by the usual form of the maximum modulus principle.

We now prove a stronger version of Theorem 7.5, where the condition (7.26) is replaced by the condition

$$\varlimsup_{r \to \infty} \frac{\ln M(r)}{r^{1/\alpha}} \leqslant 0, \tag{7.31}$$

which in turn is equivalent to the following restriction on the growth of $|f(z)|$ as $z \to \infty$: Given any $\varepsilon > 0$, there exists a sequence $\{r_n\}$ where $r_n < r_{n+1}$ $(n = 1, 2, \ldots)$ and $r_n \to \infty$ as $n \to \infty$, such that

$$|f(z)| < \exp(\varepsilon |z|^{1/\alpha}) \quad \text{if} \quad |z| = r_n \qquad (n = 1, 2, \ldots). \tag{7.32}$$

It is easy to see that (7.29) implies (7.31), since, given any $\varepsilon > 0$ and $\rho_1 < 1/\alpha$, we have

$$\exp(|z|^{\rho_1}) < \exp(\varepsilon |z|^{1/\alpha})$$

for all sufficiently large $|z|$. However, the converse is not true, as shown by Problem 7.13. Therefore the class of functions satisfying (7.32) or (7.31) is larger than the class satisfying (7.29) or (7.26).

THEOREM 7.6. *Theorem* 7.5 *remains valid if the condition* (7.26) *is replaced by* (7.31).

Proof. Suppose we know that

$$|f(x)| \leqslant C \tag{7.33}$$

at every point $z = x$ of the positive real axis. Then we can apply Theorem 7.5 to each of the two domains into which the positive real axis divides G. Each of these domains is the interior of an angle of $\pi\alpha/2$ radians, and hence, in order to use Theorem 7.5, we need only verify that (7.29) holds for all ρ_1 less than $2/\alpha$ and sufficiently close to $2/\alpha$. But (7.29) holds for all $\rho_1 \geqslant 1/\alpha$, because of (7.32).

Therefore the proof reduces to establishing the inequality (7.33) for a fixed but arbitrary point $x > 0$. Given any $\varepsilon > 0$, let $\{r_n\}$ be a sequence satisfying (7.32). As in the proof of Theorem 7.5, let G_n be the sector $|\arg z| < \alpha\pi/2$, $|z| < r_n$, and choose n so large that $x \in G_n$. The boundary of G_n consists of two line segments and an open circular arc λ_n, where

$$|f(z)| < \exp(\varepsilon r_n^{1/\alpha})$$

on λ_n, by construction, while

$$\varlimsup_{z \to \zeta} |f(z)| \leqslant C$$

for every point ζ belonging to the line segments, by hypothesis. Our strategy will be to map G_n conformally onto the interior of the unit

circle, with the point x going into the origin, and then apply the inequality (7.22), proved in connection with the two-constants theorem.

We start by making the following two consecutive transformations:

$$z_1 = f_1(z) = i\left(\frac{z}{r_n}\right)^{1/\alpha},$$

$$z_2 = f_2(z_1) = \left(\frac{1 + z_1}{1 - z_1}\right)^2.$$

The first transformation maps G_n onto the "half-disk" $D_n: |z_1| < 1$, $\operatorname{Im} z_1 > 0$, with λ_n going into the semicircle $|z_1| = 1$, $\operatorname{Im} z_1 > 0$ and x going into the point $i(x/r_n)^{1/\alpha}$. The second transformation maps D_n onto the upper half-plane, with the semicircle going into the negative real axis and the point $i(x/r_n)^{1/\alpha}$ going into the point

$$\xi_n = \left[\frac{1 + i(x/r_n)^{1/\alpha}}{1 - i(x/r_n)^{1/\alpha}}\right]^2. \tag{7.34}$$

Then we make a third transformation

$$z_3 = f_3(z_2) = \frac{z_2 - \xi_n}{z_2 - \bar{\xi}_n},$$

mapping the upper half-plane onto the interior of the unit circle $\gamma: |z_3| = 1$, with the negative real axis going into an (open) arc Σ_n with initial point 1 and final point $\xi_n/\bar{\xi}_n$ (γ is traversed in the positive direction). Thus the desired mapping of G_n onto $I(\gamma)$ is given by[7]

$$z_3 = f_3 f_2 f_1(z).$$

As a result, $f(z)$ is transformed into the function

$$f^*(z_3) = f f_1^{-1} f_2^{-1} f_3^{-1}(z_3),$$

such that $f^*(0) = f(x)$,

$$|f^*(z_3)| < \exp(\varepsilon r_n^{1/\alpha})$$

for all $z_3 \in \Sigma_n$, and

$$\overline{\lim_{z_3 \to \zeta}} |f^*(z_3)| \leqslant C$$

for all ζ in the complementary closed arc $\sigma_n = \gamma - \Sigma_n$. It follows from the inequality (7.22) that

$$|f^*(0)| = |f(x)| \leqslant [\exp(\varepsilon r_n^{1/\alpha})]^{\beta_n/\pi} C^{\alpha_n/\pi},$$

where $2\alpha_n$ and $2\beta_n$ are the angles subtended at the point $z_3 = 0$ by the arcs σ_n and Σ_n, respectively ($\alpha_n > 0$, $\beta_n > 0$, $\alpha_n + \beta_n = \pi$).

Now, according to (7.34),

$$\lim_{n \to \infty} \xi_n = 1,$$

[7] More explicitly, $f_3 f_2 f_1(z) = f_3\{f_2[f_1(z)]\}$.

which implies

$$\lim_{n\to\infty} \arg \xi_n = 0,$$

and hence, for all sufficiently large n,

$$\arg \xi_n = 4 \arg \left[1 + i\left(\frac{x}{r_n}\right)^{1/\alpha}\right] = 4 \arctan \left(\frac{x}{r_n}\right)^{1/\alpha}$$

and

$$\arg \frac{\xi_n}{\bar{\xi}_n} = 2 \arg \xi_n = 8 \arctan \left(\frac{x}{r_n}\right)^{1/\alpha} > 0.$$

Moreover, 1 is the initial point and $\xi_n/\bar{\xi}_n$ is the final point of the arc Σ_n, and therefore

$$2\beta_n = 8 \arctan \left(\frac{x}{r_n}\right)^{1/\alpha}, \qquad \text{i.e.,} \quad \beta_n = 4 \arctan \left(\frac{x}{r_n}\right)^{1/\alpha},$$

and

$$\alpha_n = \pi - 4 \arctan \left(\frac{x}{r_n}\right)^{1/\alpha}.$$

But then

$$|f(x)| \leqslant \exp\left[\frac{4}{\pi} \varepsilon r_n^{1/\alpha} \arctan \left(\frac{x}{r_n}\right)^{1/\alpha}\right] C^{1-(4/\pi)\arctan (x/r_n)^{1/\alpha}},$$

so that, passing to the limit $n \to \infty$, we obtain

$$|f(x)| \leqslant C \exp\left(\frac{4}{\pi} \varepsilon x^{1/\alpha}\right).$$

Since $\varepsilon > 0$ is arbitrary, this implies (7.33), and the proof is complete.

Remark. Theorem 7.6 cannot be strengthened, in the sense that the theorem is no longer true if we replace (7.26) by the condition

$$\lim_{r\to\infty} \frac{\ln M(r)}{r^{1/\alpha}} > 0. \tag{7.35}$$

To see this, consider the function

$$f(z) = \exp(\varepsilon z^{1/\alpha}) \qquad (\varepsilon > 0),$$

which obviously satisfies (7.35). On the boundary of the domain G, we have

$$|\arg z| = \frac{\alpha\pi}{2},$$

and hence

$$|f(re^{\pm i\alpha\pi/2})| = \exp\left(\varepsilon r^{1/\alpha} \cos\frac{\pi}{2}\right) = 1.$$

On the other hand, $|f(z)| \to \infty$ along every ray

$$\arg z = \theta, \qquad |\theta| < \frac{\alpha\pi}{2}.$$

THEOREM 7.7 (*Phragmén-Lindelöf theorem for a strip*). *Let D be the strip*

$$-\frac{h\pi}{2} < y < \frac{h\pi}{2}$$

of width h, with boundary Γ, and let $f(z)$ be analytic on D. Suppose $f(z)$ satisfies the following conditions:

1. *For every finite point $\zeta \in \Gamma$,*

$$\overline{\lim_{z \to \zeta}} |f(z)| \leqslant C < \infty;$$

2. $$\overline{\lim_{x \to -\infty}} |f(x + iy)| \leqslant C;$$

3. $$\lim_{x \to +\infty} \frac{\ln \mu(x)}{e^{x/h}} \leqslant 0,$$

where

$$\mu(x) = \sup_{-h\pi/2 < y < h\pi/2} |f(x + iy)|.$$

Then

$$|f(z)| \leqslant C$$

for all $z \in D$. Moreover, if $|f(z_0)| = C$ for some point $z_0 \in D$, then

$$f(z) \equiv C.$$

Proof. The function

$$z' = e^{z/h}$$

maps the strip D onto the right half-plane G (the interior of an angle of π radians), with the left and right-hand ends of the strip ($x = -\infty$ and $x = +\infty$) going into the points $z' = 0$ and $z' = \infty$, respectively, and the segments of the lines $x = \text{const}$ going into semicircles with common center $z' = 0$. Writing

$$f(\ln hz') = F(z'),$$

we see that $F(z')$ is analytic on G and satisfies the inequality

$$\overline{\lim_{z' \to \zeta'}} |F(z')| \leqslant C$$

at every finite point ζ' of the boundary of G. Moreover, if $r' = |z'|$ and $x = \operatorname{Re} z$, where $r' = e^{x/h}$, then

$$M(r') = \sup_{|z'| = r'} |F(z')| = \sup_{\operatorname{Re} z = x} |f(z)| = \mu(x),$$

and hence

$$\frac{\ln M(r')}{r'} = \frac{\ln \mu(x)}{e^{x/h}}.$$

It follows that

$$\lim_{r' \to \infty} \frac{\ln M(r')}{r'} = \lim_{x \to +\infty} \frac{\ln \mu(x)}{e^{x/h}} \leqslant 0.$$

Applying Theorem 7.6 to the domain G (with $\alpha = 1$), we find that

$$|F(z')| \leqslant C$$

for all $z' \in G$, which implies

$$|f(z)| \leqslant C$$

for all $z \in D$. Similarly, if $|f(z_0)| = C$ for some $z_0 \in D$, then $|F(z_0')| = C$ for $z_0' = e^{z_0/h}$, and hence $F(z') \equiv C$, which implies $f(z) \equiv C$.

PROBLEMS

7.1. If the properties listed on p. 202 are modified by replacing $-\infty$ by $+\infty$ in (7.1) and (7.2), and $\leqslant$ by $\geqslant$ in (7.3), then the function $h(z) = h(x, y)$ is said to be *superharmonic* (on G). Prove that

a) Every harmonic function is superharmonic;

b) A function is superharmonic if and only if its negative is subharmonic.

State and prove the analogue of the key lemma for superharmonic functions.

Comment. Because of part b, every result involving subharmonic functions has an analogue involving superharmonic functions.

7.2. Suppose the functions $h_1(z), \ldots, h_n(z)$ are all subharmonic on a given domain G. Prove that the linear combination

$$c_1 h_1(z) + \cdots + c_n h_n(z) \qquad (c_1 > 0, \ldots, c_n > 0)$$

and the function

$$\max \{h_1(z), \ldots, h_n(z)\}$$

are also subharmonic on G. State and prove the analogous results for superharmonic functions.

7.3. Prove that if $f(z)$ is analytic on a domain G, then the function

$$|f(z)|^p \qquad (p > 0)$$

and

$$|z|^c |f(z)|^p \qquad (c > 0, p > 0)$$

are subharmonic on G. Prove that

$$|z|^c |f(z)|^p \qquad (c < 0, p > 0)$$

is subharmonic on G if the point $z = 0$ does not belong to G.

7.4. Prove that if $f)z)$ is analytic on a domain G, and if

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\varphi})| \, d\varphi$$

for some $z_0 \in G$ and $\rho > 0$, then $f(z) \equiv$ const.

Hint. The integral

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\varphi})| \, d\varphi$$

is a strictly increasing function of ρ if $f(z) \not\equiv \text{const}$. To see this, let

$$A(\varphi)f(z_0 + \rho_1 e^{i\varphi}) = |f(z_0 + \rho_1 e^{i\varphi})| \qquad (0 \leqslant \varphi \leqslant 2\pi),$$

and use the maximum modulus principle to find the maximum of the function

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} A(\varphi)f(z_0 + ze^{i\varphi})\,d\varphi$$

on the disk $|z| \leqslant \rho_2$, where $0 < \rho_1 < \rho_2 < \Delta(z_0)$ [$\Delta(z_0)$ is the distance between z_0 and the boundary of G].

7.5. Prove that a necessary and sufficient condition for a continuous function $u(x, y)$ to be harmonic on a domain G is that

$$u(x_0, y_0) = \frac{1}{\pi\rho^2}\iint\limits_{\overline{I(\gamma_\rho)}} u(x, y)\,dx\,dy$$

for every point $z_0 = (x_0, y_0) \in G$ and all sufficiently small ρ, where γ_ρ is the circle $|z - z_0| = \rho$.

7.6. Prove that a necessary and sufficient condition for a function to be a harmonic polynomial (see p. 146) of degree less than n is that its value at every point z_0 be the average of its values at the vertices of every regular n-gon inscribed in the circle $|z| = \rho$, for all sufficiently small ρ.

7.7. Prove that a function $h(z)$ which satisfies the first two properties listed on p. 202 is subharmonic if and only if the following condition is satisfied:

3′. Given any closed Jordan curve γ such that $\overline{I(\gamma)} \subset G$ and any function $u(z)$ harmonic on $I(\gamma)$ and continuous on $\overline{I(\gamma)}$ such that $u(z) \geqslant h(z)$ for all $z \in \gamma$, then $u(z) \geqslant h(z)$ for all $z \in \overline{I(\gamma)}$.

Comment. Thus a subharmonic function can be *defined* as a function satisfying properties 1 and 2, p. 202 and property 3′ above.

7.8. Prove that if $h(z) = h(x, y)$ is subharmonic on a domain G in the z-plane and if $w = f(z)$ maps G conformally onto a domain D in the w-plane, then

$$h^*(w) = h[f^{-1}(w)]$$

is subharmonic on D.

Hint. Use the lemma on p. 165.

7.9. Give an elementary proof of Hadamard's three-circles theorem (Theorem 7.3) for the case where $f(z)$ is analytic on a closed annulus $\bar{D}: r_1 \leqslant |z| \leqslant r_2$.

Hint. The function $g(z) = z^\alpha f(z)$, where α is real, must take its maximum on the boundary of $\bar{D}$, i.e., on one of the circles $|z| = r_1$, $|z| = r_2$. Therefore

$$\rho^\alpha M(\rho) \leqslant \max\{r_1^\alpha M(r_1), r_2^\alpha M(r_2)\} \qquad (r_1 < \rho < r_2),$$

where

$$M(\rho) = \max_{|z|=\rho} |f(z)|.$$

Now choose α so that $r_1^\alpha M(r_1) = r_2^\alpha M(r_2)$.

Comment. Note that the fact that $g(z)$ is multiple-valued (when α is not an integer) causes no difficulty.

7.10. A continuous real function $u(x)$ is said to be *convex* in an interval (a, b) if the inequality

$$u[(1 - \lambda)\alpha + \lambda\beta] \leqslant (1 - \lambda)u(\alpha) + \lambda u(\beta)$$

holds for arbitrary $a < \alpha < \beta < b$ and $0 \leqslant \lambda \leqslant 1$. Verify that this is another version of the definition of a convex function given on p. 210. Prove that

a) The assumption of continuity is redundant;

b) $u(x)$ cannot have a maximum in (a, b) unless $u(x) \equiv$ const;

c) A necessary and sufficient condition for a continuous function $u(x)$ to be convex in (a, b) is that

$$u(x) \leqslant \tfrac{1}{2}[u(x - \rho) + u(x + \rho)]$$

for every $x \in (a, b)$ and all sufficiently small ρ.

Clarify the remark that "subharmonic functions are a natural two-dimensional generalization of convex functions."

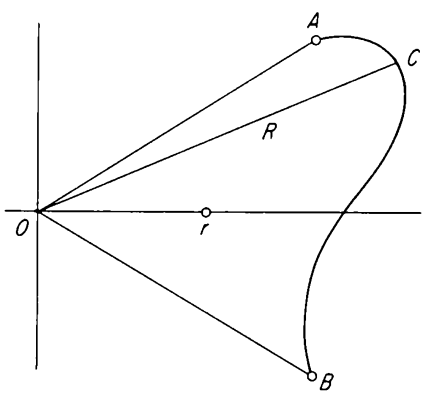

FIGURE 7.2

7.11. Let G be a Jordan domain of the type shown in Figure 7.2, whose boundary consists of two segments OA and OB of an angle OAB of $\alpha\pi$ radians, bisected by the positive real axis, together with a Jordan curve ACB which does not intersect OA or OB. Let R be the maximum distance between the origin O and a point of the curve ACB, and let $z = r$ be a point of G lying on the bisector of the angle AOB. Prove the following theorem due to Carleman: If $f(z)$ is analytic on G and if

$$\overline{\lim_{z \to \zeta}} \, |f(z)| \leqslant M$$

for every point ζ in the segments OA and OB, while

$$\overline{\lim_{z \to \zeta}} \, |f(z)| \leqslant m$$

for every point ζ in the curve ACB, then

$$|f(r)| \leqslant M^{[1-(r/R)]^{1/\alpha}} m^{(r/R)^{1/\alpha}}$$

Hint. Introduce the function

$$F_\varepsilon(z) = \exp\left[\varepsilon\left(\frac{z}{r}\right)^{1/\alpha}\right] f(z) \qquad (\varepsilon > 0),$$

and show that

$$|F_\varepsilon(z)| \leqslant \max\left\{M, m \exp\left[\varepsilon\left(\frac{R}{r}\right)^{1/\alpha}\right]\right\}.$$

Then choose ε so that

$$M = m \exp\left[\varepsilon\left(\frac{R}{r}\right)^{1/\alpha}\right].$$

7.12. Prove the following general theorem of the Phragmén-Lindelöf type: Given a domain G with boundary Γ, let $f(z)$ be analytic on G, and suppose that

$$\overline{\lim_{z \to \zeta}} \, |f(z)| \leqslant C < \infty$$

for every point $\zeta \in \Gamma$ which does not belong to some set $E \subset \Gamma$. Suppose further that there exists a function $\omega(z)$ satisfying the following conditions:

1. $\omega(z)$ is analytic on G;
2. $0 < |\omega(z)| < 1$ for all $z \in G$;
3. For every $\varepsilon > 0$,

$$\overline{\lim_{\substack{z \to \zeta \\ \zeta \in E}}} \, |\omega(z)|^{\varepsilon} |f(z)| \leqslant C.$$

Then

$$|f(z)| \leqslant C$$

for all $z \in G$. Moreover, if $|f(z_0)| = C$ for some point $z_0 \in G$, then

$$f(z) \equiv C.$$

Comment. We can allow $f(z)$ to be multiple-valued, provided $|f(z)|$ is single-valued.

7.13. Give an example of a function satisfying the condition (7.31) but not (7.26).

Ans. $f(z) = \exp\left[\dfrac{z^2}{1 + (\ln z)^2}\right]$, with $\alpha = \pi/4$.

8

THE POISSON-JENSEN FORMULA AND RELATED TOPICS

35. Various Versions of the Poisson-Jensen Formula

We begin by deriving a formula which generalizes Poisson's integral (5.17) to the case of functions with logarithmic poles (see Sec. 25):

THEOREM 8.1 (*Poisson-Jensen formula*). *Let* $u(z) = u(r, \theta)$ *be single-valued and harmonic on the disk* $r < R \leqslant \infty$, *except possibly for logarithmic poles. If* $z = 0$ *is a pole of* $u(z)$, *let its principal part be* $\mu_0 \ln |z|$; *otherwise let* $\mu_0 = 0$.[1] *Let the other poles of* $u(z)$ *be* $\zeta_1, \zeta_2, \ldots$, *arranged in order of increasing absolute value, with principal parts*

$$\mu_k \ln |z - \zeta_k| \qquad (k = 1, 2, \ldots).$$

Suppose the circle $\gamma_\rho: |z| = \rho$ *is such that* $I(\gamma_\rho)$ *contains the poles* $\zeta_1, \ldots, \zeta_{\nu(\rho)}$, *while* $E(\gamma_\rho)$ *contains the remaining poles. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} u(\rho, \varphi)\,\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\,d\varphi = u(r, \theta) - \sum_{k=0}^{\nu(\rho)} \mu_k \ln\left|\frac{\rho(z - \zeta_k)}{\rho^2 - \bar{\zeta}_k z}\right| \quad (8.1)$$

for $r < \rho < R$ *and arbitrary* θ.

[1] It is convenient to treat the logarithmic pole of $u(z)$ at the origin (if any) separately; thus, of all the numbers μ_k $(k = 0, 1, 2, \ldots)$, only μ_0 can vanish. We have already encountered a similar situation (see footnote 6, p. 54). Recall that we use the terms *increasing* and *nondecreasing* as synonyms (see Problem I.4.7).

Proof. To prove formula (8.1), called the *Poisson-Jensen formula*, consider the Möbius transformation

$$L_\zeta(z) = \rho^2 \frac{z - \zeta}{\rho^2 - z\bar{\zeta}},$$

where $|\zeta| < \rho < R$. As we know from Sec. I.49, $L_\zeta(z)$ is a conformal mapping of $I(\gamma_\rho)$ onto itself, carrying the point ζ into the center of γ_ρ [cf. (I.10.35)]. The function $L_\zeta(z)$ is analytic on the disk

$$|z| < \frac{\rho^2}{|\zeta|}, \tag{8.2}$$

has a simple zero at the point $z = \zeta$, and equals ρ for all $z \in \gamma_\rho$. Therefore

$$u_\zeta(z) = \ln \left| \frac{L_\zeta(z)}{\rho} \right|$$

is harmonic on the disk (8.2), except at the point $z = \zeta$ where it has a logarithmic pole with principal part $\ln|z - \zeta|$, and moreover $u_\zeta(z)$ vanishes for all $z \in \gamma_\rho$. But then

$$v(z) = u(z) - \sum_{k=0}^{\nu(\rho)} \mu_k u_{\zeta_k}(z) = u(z) - \sum_{k=0}^{\nu(\rho)} \mu_k \ln \left| \frac{\rho(z - \zeta_k)}{\rho^2 - \bar{\zeta}_k z} \right|$$

is harmonic on a disk of radius larger than ρ, in fact, on a disk whose radius equals the smaller of the two numbers

$$|\zeta_{\nu(\rho)+1}|, \qquad \frac{\rho^2}{|\zeta_{\nu(\rho)}|},$$

and moreover $v(z) = u(z)$ for all $z \in \gamma_\rho$. It follows from Theorem 5.5 that

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} v(\rho, \varphi) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} d\varphi,$$

which, when written out in full, is just the desired formula (8.1). If $u(z)$ has no logarithmic poles, then (8.1) obviously reduces to Poisson's integral (5.17).

COROLLARY. *Let $f(z) = f(re^{i\theta})$ be single-valued and analytic on the disk $|z| < R \leqslant \infty$, except possibly for poles. If $z = 0$ is a pole of $f(z)$, let its order be $-\lambda$, while if $z = 0$ is a zero of $f(z)$, let its order be λ; otherwise let $\lambda = 0$. Let the other zeros of $f(z)$ be $a_1, a_2, \ldots$, of orders $\alpha_1, \alpha_2, \ldots$, and let the other poles of $f(z)$ be $b_1, b_2, \ldots$, of orders $\beta_1, \beta_2, \ldots$, where both sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are arranged in order of increasing absolute value. Suppose the circle $\gamma_\rho: |z| = \rho$ is*

such that $I(\gamma_\rho)$ contains the zeros $a_1, \ldots, a_{n'(\rho)}$ and the poles $b_1, \ldots, b_{p'(\rho)}$, while $E(\gamma_\rho)$ contains the remaining zeros and poles. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})| \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\, d\varphi = \ln\left|\frac{f(re^{i\theta})}{r^\lambda}\right| + \lambda \ln \rho - \sum_{k=1}^{n'(\rho)} \alpha_k \ln\left|\frac{\rho(z - a_k)}{\rho^2 - \bar{a}_k z}\right| + \sum_{k=1}^{p'(\rho)} \beta_k \ln\left|\frac{\rho(z - b_k)}{\rho^2 - \bar{b}_k z}\right| \tag{8.3}$$

for all $z \in I(\gamma_\rho)$.

Proof. Formula (8.3), which is also known as the Poisson-Jensen formula, is an immediate consequence of Theorem 8.1. We need only observe that $\ln|f(z)|$ is single-valued and harmonic on $I(\gamma_\rho)$, except for logarithmic poles at the points $0, a_1, a_2, \ldots, b_1, b_2, \ldots$, with corresponding principal parts

$$\lambda \ln|z|, \quad \alpha_1 \ln|z - a_1|, \quad \alpha_2 \ln|z - a_2|, \ldots, \quad -\beta_1 \ln|z - b_1|, \quad -\beta_2 \ln|z - b_2|, \ldots$$

Remark 1. Suppose we agree to count each zero and pole (except at the origin) a number of times equal to its order, denoting zeros and poles by the same symbols as before, but no longer requiring that $a_j \neq a_k$, $b_j \neq b_k$ and $j \neq k$. Correspondingly, let $I(\gamma_\rho)$ contain $n(\rho)$ zeros and $p(\rho)$ poles, each counted a number of times equal to its order. Then formula (8.3) takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(re^{i\theta})| \frac{\rho^2 + r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\, d\varphi = \ln\left|\frac{f(re^{i\theta})}{r^\lambda}\right| + \lambda \ln \rho - \sum_{k=1}^{n(\rho)} \ln\left|\frac{\rho(z - a_k)}{\rho^2 - \bar{a}_k z}\right| + \sum_{k=1}^{p(\rho)} \ln\left|\frac{\rho(z - b_k)}{\rho^2 - \bar{b}_k z}\right|, \tag{8.4}$$

i.e., there is no longer any need for factors α_k and β_k as in (8.3), since each multiple zero and multiple pole is now automatically counted the correct number of times.

Remark 2. If $f(z)$ has no poles in the disk $|z| < R$, then all the β_k vanish and λ is nonnegative. Then formula (8.3) reduces to

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})| \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\, d\varphi = \ln\left|\frac{f(re^{i\theta})}{r^\lambda}\right| + \lambda \ln \rho - \sum_{k=1}^{n'(\rho)} \alpha_k \ln\left|\frac{\rho(z - a_k)}{\rho^2 - \bar{a}_k z}\right|,$$

while (8.4) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})| \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\, d\varphi = \ln\left|\frac{f(re^{i\theta})}{r^\lambda}\right| + \lambda \ln \rho - \sum_{k=1}^{n(\rho)} \ln\left|\frac{\rho(z - a_k)}{\rho^2 - \bar{a}_k z}\right|. \tag{8.5}$$

THEOREM 8.2. *Theorem* 8.1 *continues to hold if* $I(\gamma_\rho)$ *is replaced by* $\overline{I(\gamma_\rho)}$, *i.e.,* $\nu(\rho)$ *can equally well be interpreted as the number of logarithmic poles in the closed disk* $|z| \leqslant \rho$.

Proof. Suppose $u(z)$ has n logarithmic poles

$$\zeta_{\nu(\rho)+1}, \cdots, \zeta_{\nu(\rho)+n} \tag{8.6}$$

on the circle γ_ρ itself. Then the function

$$u(z) - \sum_{k=\nu(\rho)+1}^{\nu(\rho)+n} \mu_k \ln |z - \zeta_k|$$

has no logarithmic poles on γ_ρ, and the same poles (and principal parts) as $u(z)$ in $I(\gamma_\rho)$. Therefore, according to Theorem 8.1,

$$\frac{1}{2\pi}\int_0^{2\pi}\left[u(\rho, \varphi) - \sum_{k=\nu(\rho)+1}^{\nu(\rho)+n} \mu_k \ln |\rho e^{i\varphi} - \zeta_k|\right] \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}\, d\varphi$$
$$= u(r, \theta) - \sum_{k=\nu(\rho)+1}^{\nu(\rho)+n} \mu_k \ln |re^{i\theta} - \zeta_k| - \sum_{k=0}^{\nu(\rho)} \mu_k \ln \left|\frac{\rho(z - \zeta_k)}{\rho^2 - \bar{\zeta}_k z}\right|. \tag{8.7}$$

This leads us to examine separately the improper integral

$$J_{\zeta_0} = \frac{1}{2\pi}\int_0^{2\pi} \ln |\rho e^{i\varphi} - \zeta_0| \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}\, d\varphi \tag{8.8}$$
$$(\zeta_0 = \rho e^{i\varphi_0},\ r < \rho),$$

which is easily shown to be convergent (see Problem 8.1). We will evaluate J_{ζ_0} by a continuity argument. In fact, writing

$$\zeta = (\rho + \delta)e^{i\varphi_0} \qquad (0 < \delta < 1),$$

and forming the integral

$$J_\zeta = \frac{1}{2\pi}\int_0^{2\pi} \ln |\rho e^{i\varphi} - \zeta| \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}\, d\varphi \qquad (r < \rho),$$

we will show that

$$J_{\zeta_0} = \lim_{\zeta \to \zeta_0} J_\zeta. \tag{8.9}$$

But then, since $\ln |z - \zeta|$ is harmonic for $|z| < \rho + \delta$, and hence

$$J_\zeta = \ln |re^{i\theta} - \zeta|,$$

it will follow that

$$J_{\zeta_0} = \ln |re^{i\theta} - \zeta_0|. \tag{8.10}$$

To verify (8.9), consider the difference

$$\Delta = J_\zeta - J_{\zeta_0} = \frac{1}{2\pi}\int_{\varphi_0 - \pi}^{\varphi_0 + \pi} \ln \left|\frac{\rho e^{i\varphi} - \zeta}{\rho e^{i\varphi} - \zeta_0}\right| \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)}\, d\varphi$$
$$(r < \rho),$$

where the periodicity of the integrand has been used to change the limits of integration. To estimate Δ, we need the inequality

$$|\rho e^{i\varphi} - \zeta_0| < |\rho e^{i\varphi} - \zeta|.$$

If $\rho e^{i\varphi} = \zeta_0$, the inequality is obvious; if $\rho e^{i\varphi} \neq \zeta_0$, the points $\rho e^{i\varphi}$, $\zeta_0 = \rho e^{i\varphi_0}$ and $\zeta = (\rho + \delta)e^{i\varphi_0}$ are the vertices of a triangle with an obtuse angle at the vertex ζ_0, and the inequality follows at once (draw a figure). Therefore

$$0 < \ln \left| \frac{\rho e^{i\varphi} - \zeta}{\rho e^{i\varphi} - \zeta_0} \right| = \ln \left| 1 + \frac{\zeta_0 - \zeta}{\rho e^{i\varphi} - \zeta_0} \right| \leqslant \ln \left(1 + \left| \frac{\zeta_0 - \zeta}{\rho e^{i\varphi} - \zeta_0} \right| \right)$$
$$= \ln \left(1 + \frac{\delta}{2\rho \sin \frac{1}{2}|\varphi - \varphi_0|} \right) < \ln \left(1 + \frac{\pi\delta}{\rho|\varphi - \varphi_0|} \right)$$

[cf. (2.15)], and moreover

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} \leqslant \max_{0 \leqslant \theta, \varphi \leqslant 2\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} = \frac{\rho + r}{\rho - r}.$$

Thus we have

$$0 < \Delta < \frac{\rho + r}{\rho - r} \frac{1}{2\pi} \int_{\varphi_0 - \pi}^{\varphi_0 + \pi} \ln \left(1 + \frac{\pi\delta}{\rho|\varphi - \varphi_0|} \right) d\varphi$$
$$= \frac{1}{\pi} \frac{\rho + r}{\rho - r} \int_0^{\pi} \ln \left(1 + \frac{\pi\delta}{\rho\varphi} \right) d\varphi$$

(verify that this improper integral exists). Given any $\varepsilon > 0$, we can find a number α $(0 < \alpha < \pi)$ such that

$$\frac{1}{\pi} \frac{\rho + r}{\rho - r} \int_0^{\alpha} \ln \left(1 + \frac{\pi\delta}{\rho\varphi} \right) d\varphi < \frac{1}{\pi} \frac{\rho + r}{\rho - r} \int_0^{\alpha} \ln \left(1 + \frac{\pi}{\rho\varphi} \right) d\varphi < \frac{\varepsilon}{2}.$$

Moreover, holding α fixed, we have

$$\frac{1}{\pi} \frac{\rho + r}{\rho - r} \int_{\alpha}^{\pi} \ln \left(1 + \frac{\pi\delta}{\rho\varphi} \right) d\varphi < \frac{1}{\pi} \frac{\rho + r}{\rho - r} \int_{\alpha}^{\pi} \frac{\pi\delta}{\rho\varphi} d\varphi < \frac{\pi\delta}{\rho\alpha} \frac{\rho + r}{\rho - r} < \frac{\varepsilon}{2}$$

for sufficiently small $\delta < \delta_0(\varepsilon)$. It follows that

$$0 < \Delta < \varepsilon \quad \text{if} \quad \delta < \delta_0(\varepsilon),$$

which proves (8.9), and hence (8.10).

Finally, we apply (8.10) to each of the poles (8.6) on the circle γ_ρ, obtaining the formula

$$\frac{1}{2\pi} \int_0^{2\pi} \sum_{k = \nu(\rho) + 1}^{\nu(\rho) + n} \mu_k \ln |\rho e^{i\varphi} - \zeta_k| \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \varphi)} d\varphi$$
$$= \sum_{k = \nu(\rho) + 1}^{\nu(\rho) + n} \mu_k \ln |re^{i\theta} - \zeta_k|.$$

Together with (8.7), this implies

$$\frac{1}{2\pi}\int_0^{2\pi} u(\rho, \varphi)\,\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\,d\varphi = u(r, \theta) - \sum_{k=0}^{\nu(\rho)} \mu_k \ln\left|\frac{\rho(z - \zeta_k)}{\rho^2 - \bar{\zeta}_k z}\right|.$$

Thus Theorem 8.1 remains valid if $E(\gamma_\rho)$ is replaced by $\overline{E(\gamma_\rho)}$, i.e., some of the poles not in $I(\gamma_\rho)$ can lie on γ_ρ. However, we might just as well interpret $\nu(\rho)$ as the number of poles in $\overline{I(\gamma_\rho)}$, since

$$\ln\left|\frac{\rho(z - \zeta_k)}{\rho^2 - \bar{\zeta}_k z}\right| = \ln\left|\frac{\rho}{\bar{\zeta}_k}\,\frac{z - \zeta_k}{\zeta_k - z}\right| = 0$$

for any $\zeta_k \in \gamma_\rho$, so that including these poles in the sum in the right-hand side of (8.1) does not affect the result. This completes the proof.

Remark. Obviously, formulas (8.3), (8.4) and (8.5) can be similarly reinterpreted to permit zeros and poles on the circle γ_ρ. From now on, we shall make this interpretation. In particular, $n(\rho)$ will denote the number of zeros in $\overline{I(\gamma_\rho)}$ and $p(\rho)$ the number of poles in $\overline{I(\gamma_\rho)}$, each counted a number of times equal to its order.

36. Jensen's Inequality. Blaschke Products

We now proceed to derive a number of consequences of the Poisson-Jensen formula:

THEOREM 8.3. *Let $f(z)$ be analytic on the disk $|z| < R$.*[2] *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})|\,d\varphi$$

is an increasing function of ρ $(0 \leqslant \rho < R)$.

Proof. Noting that

$$\frac{f(z)}{z^\lambda} = \frac{f^{(\lambda)}(0)}{\lambda!} + \frac{f^{(\lambda+1)}(0)}{(\lambda+1)!}\,z + \cdots$$

if $f(z)$ has a zero of order λ at the origin, and setting $z = 0$ (i.e., $r = 0$) in formula (8.5), we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})|\,d\varphi = \ln\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right| + \lambda\ln\rho - \sum_{k=1}^{n(\rho)} \ln\frac{|a_k|}{\rho}$$

[2] We revert to our convention that all functions are single-valued, unless the contrary is explicitly stated.

or

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})|\,d\varphi = \ln\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right| + \ln\left(\rho^{\lambda}\prod_{k=1}^{n(\rho)}\frac{\rho}{|a_k|}\right). \tag{8.11}$$

Obviously the right-hand side of (8.11) increases with ρ, since each factor

$$\frac{\rho}{|a_k|} \geqslant 1$$

in the product increases with ρ, while the number of factors itself increases for certain values of ρ, namely those such that the circle $|z| = \rho$ passes through zeros of $f(z)$. Therefore the left-hand side of (8.11) is also an increasing function of ρ, as asserted.

COROLLARY 1. *Let $f(z)$ be analytic on the disk $|z| < R$, and let*

$$M(\rho) = \max_{0 \leqslant \varphi \leqslant 2\pi} |f(\rho e^{i\varphi})| \qquad (0 < \rho < R).$$

Then

$$\frac{\rho^{n(\rho)}}{|a_1|\cdots|a_{n(\rho)}|} \leqslant \frac{M(\rho)}{\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right|\rho^{\lambda}}, \tag{8.12}$$

where the notation is the same as in (8.5).

Proof. Formula (8.12), known as *Jensen's inequality*, is another way of writing

$$\ln\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right| + \ln\left(\rho^{\lambda}\prod_{k=1}^{n(\rho)}\frac{\rho}{|a_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi}\ln|f(\rho e^{i\varphi})|\,d\varphi \leqslant \ln M(\rho).$$

COROLLARY 2. *If $f(z)$ and $M(\rho)$ are the same as in Corollary* 1, *then*

$$\int_0^{\rho}\frac{n(r)}{r}\,dr \leqslant \ln\frac{M(\rho)}{\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right|\rho^{\lambda}}. \tag{8.13}$$

Proof. Taking the logarithm of both sides of the identity

$$\frac{\rho^{n(\rho)}}{|a_1|\cdots|a_{n(\rho)}|} = \frac{|a_2|}{|a_1|}\,\frac{|a_3|^2}{|a_2|^2}\cdots\frac{|a_{n(\rho)}|^{n(\rho)-1}}{|a_{n(\rho)-1}|^{n(\rho)-1}}\,\frac{\rho^{n(\rho)}}{|a_{n(\rho)}|^{n(\rho)}},$$

we obtain

$$\begin{aligned}\ln\frac{\rho^{n(\rho)}}{|a_1|\cdots|a_{n(\rho)}|} &= \ln\left|\frac{a_2}{a_1}\right| + 2\ln\left|\frac{a_3}{a_2}\right| + \cdots \\ &\quad + [n(\rho)-1]\ln\left|\frac{a_{n(\rho)}}{a_{n(\rho)-1}}\right| + n(\rho)\ln\left|\frac{\rho}{a_{n(\rho)}}\right| \\ &= \int_{|a_1|}^{|a_2|}\frac{dr}{r} + \int_{|a_2|}^{|a_3|}\frac{2}{r}\,dr + \cdots \\ &\quad + \int_{|a_{n(\rho)-1}|}^{|a_{n(\rho)}|}\frac{n(\rho)-1}{r}\,dr + \int_{|a_{n(\rho)}|}^{\rho}\frac{n(\rho)}{r}\,dr\end{aligned} \tag{8.14}$$

But the numerical factor in the numerator of the integral from $|a_k|$ to $|a_{k+1}|$ is just the value of the function $n(r)$ in the interval $|a_k| \leqslant r < |a_{k+1}|$.[3] Therefore (8.14) becomes

$$\ln \frac{\rho^{n(\rho)}}{|a_1| \cdots |a_{n(\rho)}|} = \int_{|a_1|}^{|a_2|} \frac{n(r)}{r}\, dr + \int_{|a_2|}^{|a_3|} \frac{n(r)}{r}\, dr + \cdots + \int_{|a_{n(\rho)-1}|}^{|a_{n(\rho)}|} \frac{n(r)}{r}\, dr + \int_{|a_{n(\rho)}|}^{\rho} \frac{n(r)}{r}\, dr = \int_{|a_1|}^{\rho} \frac{n(r)}{r}\, dr.$$

Since $n(r) = 0$ for $r < |a_1|$,[4] we can set the lower limit of integration in the last integral equal to zero. This gives

$$\ln \frac{\rho^{n(\rho)}}{|a_1| \cdots |a_{n(\rho)}|} = \int_0^{\rho} \frac{n(r)}{r}\, dr,$$

and (8.13) then follows from Jensen's inequality in the form (8.12).

Remark. When written in the form (8.13), Jensen's inequality expresses the relation between the growth of $n(\rho)$, the number of zeros of $f(z)$ in the disk $|z| \leqslant \rho$, and the growth of the maximum modulus function $M(\rho)$.

We now use formula (8.11) to study certain functions analytic on a finite disk $K: |z| < R$. It follows from Theorem 8.3 that the limit[5]

$$\lim_{\rho \to R} \frac{1}{2\pi} \int_0^{2\pi} \ln |f(\rho e^{i\varphi})|\, d\varphi \tag{8.15}$$

always exists, whether finite or infinite. Consider the class of functions satisfying the condition

$$\lim_{\rho \to R} \frac{1}{2\pi} \int_0^{2\pi} \ln |f(\rho e^{i\varphi})|\, d\varphi < \infty. \tag{8.16}$$

Such functions are called *functions of bounded logarithmic mean.* For example, (8.16) is satisfied if $f(z)$ is bounded on K, i.e., if

$$|f(z)| \leqslant M < \infty \qquad (z \in K),$$

since then the limit (8.15) does not exceed $\ln M$.[6] However, the condition (8.16) is also satisfied by a large class of unbounded functions. In fact, the restriction represented by (8.16) involves only the distribution of zeros of

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[3] Note that $n(r)$ is constant in this (half-open) interval.

[4] Recall that $n(\rho)$, the number of zeros in the disk $|z| \leqslant \rho$, does not include a possible zero at the origin.

[5] Strictly speaking, the limit as $\rho \to R-$.

[6] In particular, (8.16) is satisfied for $f(z) \equiv 0$, since $-\infty < +\infty$.

$f(z)$, as shown by

THEOREM 8.4. *The condition* (8.16) *holds if* $f(z)$ *has finitely many zeros in the disk* $K: |z| < R$. *If* $f(z) \not\equiv 0$ *has infinitely many zeros in* K, *then* (8.16) *is equivalent to convergence of the infinite product*

$$\prod_{k=1}^{\infty} \frac{R}{|a_k|}. \tag{8.17}$$

Proof. According to (8.11), the condition (8.16) holds if and only if

$$\ln \prod_{k=1}^{n(\rho)} \frac{\rho}{|a_k|} \leqslant \text{const}$$

for all $\rho < R$. But this is equivalent to the condition

$$\prod_{k=1}^{n(\rho)} \frac{\rho}{|a_k|} \leqslant \text{const} = C > 0$$

for all $\rho < R$. The first assertion of the theorem follows at once from the observation that

$$\prod_{k=1}^{n} \frac{\rho}{|a_k|} < \prod_{k=1}^{n} \frac{R}{|a_k|} \leqslant C$$

for all $\rho < R$, if there are finitely many (say n) zeros in K. Thus we need only consider the case of infinitely many zeros in K. Suppose (8.17) converges. Then

$$\prod_{k=1}^{n(\rho)} \frac{\rho}{|a_k|} < \prod_{k=1}^{n(\rho)} \frac{R}{|a_k|} < \prod_{k=1}^{\infty} \frac{R}{|a_k|} = C$$

for all $\rho < R$. Conversely, if

$$\prod_{k=1}^{n(\rho)} \frac{\rho}{|a_k|} \leqslant C$$

for all $\rho < R$, then, given any integer $n > 0$, we have

$$\prod_{k=1}^{n} \frac{\rho}{|a_k|} \leqslant \prod_{k=1}^{n(\rho)} \frac{\rho}{|a_k|} \leqslant C$$

for all $n(\rho) > n$ [note that $n(\rho) \to \infty$ as $\rho \to R$], i.e.,

$$\prod_{k=1}^{n} \frac{\rho}{|a_k|} \leqslant C$$

for all n. Taking the limit as $\rho \to R$, we find that

$$\prod_{k=1}^{n} \frac{R}{|a_k|} \leqslant C$$

for all n, and hence (8.17) converges, as asserted.

COROLLARY 1. *If* $f(z) \not\equiv 0$ *has infinitely many zeros in* K, *then the condition* (8.16) *is equivalent to convergence of the infinite series*

$$\sum_{k=1}^{\infty} (R - |a_k|). \tag{8.18}$$

Proof. Writing

$$\frac{R}{|a_k|} = 1 + \frac{R - |a_k|}{|a_k|},$$

and applying Theorem I.15.14, we see that (8.17) converges if and only if

$$\sum_{k=1}^{\infty} \frac{R - |a_k|}{|a_k|} \tag{8.19}$$

converges. But $|a_k| \to R$ as $k \to \infty$ if $f(z)$ has infinitely many zeros in K, since otherwise $f(z)$ would have a limit point of zeros at an interior point of K, which would imply that $f(z) \equiv 0$ (why?), contrary to hypothesis. Therefore, starting from some value of k, we have

$$\frac{R}{2} < |a_k|,$$

and hence

$$\frac{1}{R}(R - |a_k|) < \frac{R - |a_k|}{|a_k|} < \frac{2}{R}(R - |a_k|).$$

It follows that convergence of (8.19) is equivalent to convergence of (8.18) [the latter represents the sum of the distances from the zeros a_k to the circle $|z| = R$].[7]

COROLLARY 2. *If* $f(z)$ *satisfies the condition* (8.16), *and if* $f(z)$ *vanishes at all points of a sequence* $\{\zeta_k\}$ *such that the series*

$$\sum_{k=1}^{\infty} (R - |\zeta_k|) \tag{8.20}$$

diverges, then $f(z) \equiv 0$.

Proof. If $f(z) \not\equiv 0$, then the series (8.18) converges, and hence so does the series (8.20), since $\{\zeta_k\}$ is a subsequence of $\{a_k\}$. This contradiction shows that $f(z) \equiv 0$.

---

[7] Regarding all products and series with only a finite number of terms as "convergent," we can say that the condition (8.16) holds if and only if the product and series

$$\prod_k \frac{R}{|a_k|}, \qquad \sum_k (R - |a_k|)$$

converge. Here we deliberately omit the upper limit (which may be finite), and simplify the lower limit.

Corollary 2 immediately raises the following question: If $f(z)$ and $g(z)$ are two functions which are analytic on the disk $K: |z| < R$ and satisfy the condition (8.16), and if $f(z)$ and $g(z)$ coincide at all points of a sequence $\{\zeta_k\}$ such that the series (8.20) diverges, then do $f(z)$ and $g(z)$ coincide at all points of K? Obviously, the answer to this question would be in the affirmative, if it were known that the difference $f(z) - g(z)$ satisfies (8.16) whenever $f(z)$ and $g(z)$ satisfy (8.16). However, a simple example shows that this is not the case, and hence the conjectured uniqueness theorem is false. In fact, let

$$f(z) = \exp\left(\frac{\pi i}{1 - z}\right), \qquad g(z) = \exp\left(\frac{\pi i}{z - 1}\right).$$

Then both functions are analytic and do not vanish in the unit disk $K: |z| < 1$. Because of the absence of zeros in K, each of the functions satisfies a condition of the form (8.16). Their difference

$$f(z) - g(z) = 2i \sin \frac{\pi}{1 - z} \tag{8.21}$$

is also analytic on K, but has zeros at the points

$$\zeta_k = 1 - \frac{1}{k} \qquad (k = 1, 2, \ldots),$$

for which the series

$$\sum_{k=1}^{\infty} (1 - |\zeta_k|) = \sum_{k=1}^{\infty} \left[1 - \left(1 - \frac{1}{k}\right)\right] = \sum_{k=1}^{\infty} \frac{1}{k}$$

obviously diverges. Since (8.21) is not identically zero on K, it follows that (8.21) does not satisfy the condition (8.16). In other words, the class of functions analytic on K and satisfying (8.16) is not closed under addition (subtraction). In the next section, this lack of closure will be remedied by suitably restricting the class of functions satisfying (8.16).

According to Theorem 8.4, Corollary 2, given a function $f(z)$ analytic on K with zeros $\{a_k\}$ in K, the condition (8.16) is equivalent to convergence of the series (8.18). Conversely, given a sequence $\{\zeta_k\}$ such that the series (8.20) converges, does there always exist a function $f(z)$ analytic on K with zeros $\{\zeta_k\}$ in K? The answer to this question is in the affirmative, as shown by

THEOREM 8.5. *Given any nonnegative integer* λ *and any increasing sequence*[8] *of nonzero complex numbers* $\{\zeta_k\}$ *in the disk* $K: |z| < R$ *such that*

$$\sum_{k=1}^{\infty} (R - |\zeta_k|) < \infty,$$

[8] I.e., a sequence arranged in order of increasing absolute value ($|\zeta_k| \leqslant |\zeta_{k+1}|$).

there exists a function $b(z)$ which is analytic and bounded by 1 on K, whose zeros in K coincide with the points

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, \zeta_1, \ldots, \zeta_k, \ldots. \tag{8.22}$$

Proof. Consider the product

$$b_n(z) = \left(\frac{z}{R}\right)^{\lambda} \prod_{k=1}^{n} \frac{R|\zeta_k|}{\zeta_k} \frac{\zeta_k - z}{R^2 - \bar{\zeta}_k z}. \tag{8.23}$$

Obviously $b_n(z)$ has zeros at the $\lambda + n$ points

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, \zeta_1, \ldots, \zeta_n,$$

and only at these points. Moreover $b_n(z)$ is analytic on K and continuous on $\bar{K}$, while $|b_n(z)| = 1$ on the circle $|z| = R$. Therefore

$$|b_n(z)| < 1 \tag{8.24}$$

for all n and $z \in K$ (why?). Clearly, (8.23) is a partial product of the infinite product

$$\left(\frac{z}{R}\right)^{\lambda} \prod_{k=1}^{\infty} \frac{R|\zeta_k|}{\zeta_k} \frac{\zeta_k - z}{R^2 - \bar{\zeta}_k z}, \tag{8.25}$$

whose general term can be written in the form

$$1 - \frac{R\zeta_k + z|\zeta_k|}{\zeta_k(R^2 - \bar{\zeta}_k z)}(R - |\zeta_k|) = 1 + \lambda_k(R - |\zeta_k|).$$

But

$$|\lambda_k| = \left|\frac{R\zeta_k + z|\zeta_k|}{\zeta_k(R^2 - \bar{\zeta}_k z)}\right| \leqslant \frac{R|\zeta_k| + r|\zeta_k|}{|\zeta_k|(R^2 - |\zeta_k| r)} < \frac{R + r}{R^2 - Rr} = \frac{R + r}{R(R - r)}$$

if $|z| \leqslant r < R$. Therefore, since

$$\sum_{k=1}^{\infty} (R - |\zeta_k|)$$

is convergent, the series

$$\sum_{k=1}^{\infty} |\lambda_k(R - |\zeta_k|)|$$

is uniformly convergent on every compact subset of K. It follows from an obvious modification of Theorem I.15.16 and the last sentence of Sec. I.77 that the infinite product (8.25) converges to a function $b(z)$ analytic on K, whose zeros coincide with (8.22). The function $b(z)$ is called the *Blaschke product* corresponding to the zeros (8.22). Finally, we note that (8.24), together with

$$b(z) = \lim_{n \to \infty} b_n(z),$$

implies that $|b(z)| \leqslant 1$ for all $z \in K$, as asserted.

COROLLARY. *Let $f(z)$ be analytic and of bounded logarithmic mean on the disk $K: |z| < R$, and suppose the zeros of $f(z)$ are given by the increasing sequence*

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, a_1, \ldots, a_k, \ldots.$$

Then $f(z)$ can be represented in the form

$$f(z) = F(z)b(z) = F(z)\left(\frac{z}{R}\right)^{\lambda} \prod_{k=1}^{\infty} \frac{R|a_k|}{a_k} \frac{a_k - z}{R^2 - \bar{a}_k z}, \tag{8.26}$$

where $F(z)$ is analytic and nonvanishing on K, and $b(z)$ is analytic and bounded by 1 on K.

Proof. The function $b(z)$ is the Blaschke product corresponding to the zeros of $f(z)$.[9]

37. Functions of Bounded Characteristic

Given a real function $h(z)$ defined on a set E, let $h^+(z)$ denote the function equal to $h(z)$ at the points where $h(z) \geqslant 0$ and equal to zero where $h(z) < 0$, i.e.,

$$h^+(z) = \max\{0, h(z)\}$$

or

$$h^+(z) = \tfrac{1}{2}\{h(z) + |h(z)|\}.$$

Similarly, let $h^-(z)$ denote the function

$$h^-(z) = \max\{0, -h(z)\}$$

or

$$h^-(z) = \tfrac{1}{2}\{-h(z) + |h(z)|\},$$

equal to $-h(z)$ if $h(z) \leqslant 0$ and zero if $h(z) > 0$. Clearly we have

$$h(z) = h^+(z) - h^-(z), \qquad |h(z)| = h^+(z) + h^-(z).$$

If $h(z)$ is subharmonic on a domain G, then so is the function $h^+(z)$. In fact,

$$h(z_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + \rho e^{i\varphi})\, d\varphi \leqslant \frac{1}{2\pi}\int_0^{2\pi} h^+(z_0 + \rho e^{i\varphi})\, d\varphi \tag{8.27}$$

for every point $z_0 \in G$ and all sufficiently small ρ. If $h(z_0) \geqslant 0$, then $h^+(z_0) = h(z_0)$ and (8.27) implies

$$h^+(z_0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} h^+(z_0 + \rho e^{i\varphi})\, d\varphi. \tag{8.28}$$

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[9] Formula (8.26) is the analogue of the familiar factorization of a polynomial into linear factors [cf. (I.9.2)].

Moreover, (8.28) obviously remains true if $h(z_0) < 0$, since then $h^+(z_0) = 0$, while the right-hand side is nonnegative. It follows that if $f(z)$ is analytic on a domain G, then $\ln^+ |f(z)|$ is subharmonic on G.

Now suppose $f(z) \not\equiv 0$ is analytic on the disk $|z| < R$. According to (8.11),

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi = \ln\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right| + \ln \sum_{k=1}^{n(\rho)} \frac{\rho}{|a_k|} + \lambda \ln \rho + \frac{1}{2\pi}\int_0^{2\pi} \ln^- |f(\rho e^{i\varphi})|\, d\varphi,$$

which implies

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^- |f(\rho e^{i\varphi})|\, d\varphi \leqslant \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi - \ln\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right| - \lambda \ln \rho$$

(why?). Therefore, if the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi$$

is bounded for all ρ in the interval $(0, R)$, then so are the integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^- |f(\rho e^{i\varphi})|\, d\varphi$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} |\ln |f(\rho e^{i\varphi})|\,|\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi + \frac{1}{2\pi}\int_0^{2\pi} \ln^- |f(\rho e^{i\varphi})|\, d\varphi.$$

Moreover, it is obvious that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi \leqslant \frac{1}{2\pi}\int_0^{2\pi} |\ln |f(\rho e^{i\varphi})|\,|\, d\varphi.$$

Thus we see at once that the conditions

$$\overline{\lim_{\rho \to R}} \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi < \infty \tag{8.29}$$

and

$$\overline{\lim_{\rho \to R}} \frac{1}{2\pi}\int_0^{2\pi} |\ln |f(\rho e^{i\varphi})|\,|\, d\varphi < \infty \tag{8.30}$$

are equivalent (see Problem 8.9).

A function $f(z) \not\equiv 0$ which is analytic on the disk $|z| < R$ and satisfies the condition (8.29), or the equivalent condition (8.30), is called a *function of bounded characteristic*, and the class of all such functions will be denoted by N. Since

$$\lim_{\rho \to R} \frac{1}{2\pi}\int_0^{2\pi} \ln |f(\rho e^{i\varphi})|\, d\varphi \leqslant \overline{\lim_{\rho \to R}} \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi,$$

every function of bounded characteristic is a function of bounded logarithmic mean (see p. 231). However, the converse is not true, as shown by the function

$$f(z) = \exp\left(\frac{\pi i}{z-1}\right)$$

already considered on p. 234. In fact, since $f(z)$ is analytic and nonvanishing on the unit disk $K: |z| < 1$, the function

$$\ln|f(z)| = \frac{\pi\rho \sin\varphi}{1 + \rho^2 - 2\rho\cos\varphi}$$

is harmonic on K. Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})|\, d\varphi = \ln|f(0)| = 0$$

for all ρ in $(0, R)$, and hence

$$\lim_{\rho \to R} \frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})|\, d\varphi = 0 < \infty,$$

i.e., $f(z)$ is a function of bounded logarithmic mean. On the other hand, since

$$\ln^+|f(\rho e^{i\varphi})| = \begin{cases} \ln|f(\rho e^{i\varphi})| & \text{if } 0 \leqslant \varphi \leqslant \pi, \\ 0 & \text{if } \pi < \varphi \leqslant 2\pi, \end{cases}$$

it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+|f(\rho e^{i\varphi})|\, d\varphi = \frac{1}{2\pi}\int_0^{\pi} \frac{\pi\rho\sin\varphi}{1+\rho^2-2\rho\cos\varphi}\, d\varphi$$
$$= \frac{1}{4}\ln(1+\rho^2-2\rho\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi} = \frac{1}{2}\ln\frac{1+\rho}{1-\rho} \to \infty \qquad \text{as} \quad \rho \to 1,$$

i.e., $f(z)$ is not a function of bounded characteristic.

We now show that the class N has the algebraic closure discussed on p. 234, which is lacking in the case of the class of functions of bounded logarithmic mean:

THEOREM 8.6. *The class N is a ring of functions, i.e., if $f(z)$ and $g(z)$ belong to N, then so do the functions $f(z) \pm g(z)$ and $f(z)g(z)$.*[10]

Proof. The theorem is trivial if $f(z) \pm g(z) \equiv 0$, or $f(z)g(z) \equiv 0$, since N contains the function which is identically zero. Therefore we can assume that $f(z) \pm g(z) \not\equiv 0$, or $f(z)g(z) \not\equiv 0$. If $|f(z)| \leqslant 1$ and $|g(z)| \leqslant 1$, then

$$\ln^+(|f(z)| + |g(z)|) \leqslant \ln 2,$$

[10] This again shows that N is a proper subset of the class of functions of bounded logarithmic mean.

while if at least one of the numbers $|f(z)|$ and $|g(z)|$ is >1, we have

$$\begin{aligned}\ln^+ &(|f(z)| + |g(z)|) \\ &= \ln(|f(z)| + |g(z)|) \leqslant \ln[2\max\{|f(z)|, |g(z)|\}] \\ &= \ln 2 + \ln[\max\{|f(z)|, |g(z)|\}] \leqslant \ln 2 + \ln^+ |f(z)| + \ln^+ |g(z)|.\end{aligned}$$

Thus, in any event,

$$\begin{aligned}\ln^+ |f(z) \pm g(z)| &\leqslant \ln^+ (|f(z)| + |g(z)|) \\ &\leqslant \ln 2 + \ln^+ |f(z)| + \ln^+ |g(z)|,\end{aligned}$$

and hence

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} &\ln^+ |f(\rho e^{i\varphi}) \pm g(\rho e^{i\varphi})|\, d\varphi \\ &\leqslant \ln 2 + \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi + \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |g(\rho e^{i\varphi})|\, d\varphi.\end{aligned}$$

Therefore, as asserted, $f(z) \pm g(z) \in N$ whenever $f(z) \in N$, $g(z) \in N$.

Similarly, if $|f(z)| \geqslant 1$ and $|g(z)| \geqslant 1$, then

$$\begin{aligned}\ln^+ |f(z)g(z)| = \ln |f(z)g(z)| &= \ln |f(z)| + \ln |g(z)| \\ &= \ln^+ |f(z)| + \ln^+ |g(z)|,\end{aligned}$$

while if at least one of the numbers $|f(z)|$ and $|g(z)|$ is <1, we have

$$\ln^+ |f(z)g(z)| \leqslant \ln^+ [\max\{|f(z)|, |g(z)|\}] \leqslant \ln^+ |f(z)| + \ln^+ |g(z)|.$$

Thus, in any event,

$$\ln^+ |f(z)g(z)| \leqslant \ln^+ |f(z)| + \ln^+ |g(z)|,$$

and hence

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} &\ln^+ |f(\rho e^{i\varphi})g(\rho e^{i\varphi})|\, d\varphi \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi + \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |g(\rho e^{i\varphi})|\, d\varphi,\end{aligned}$$

i.e., $f(z)g(z) \in N$ whenever $f(z) \in N$, $g(z) \in N$, and the proof is complete.

COROLLARY (*Uniqueness theorem for the class N*). *If $f(z)$ and $g(z)$ are two functions in N which coincide at all points of a sequence $\{\zeta_n\}$ such that the series*

$$\sum_{n=1}^{\infty} (R - |\zeta_n|) \qquad (|\zeta_n| < R) \tag{8.31}$$

diverges, then $f(z) \equiv g(z)$.

Proof. The result is an immediate consequence of Theorem 8.4, Corollary 2, since $f(z) - g(z)$ belongs to the class N, and *a fortiori* to the class of functions of bounded logarithmic mean.

Remark 1. As already noted on p. 234, the corollary cannot be extended to the class of functions of bounded logarithmic mean, which is not closed under addition.

Remark 2. If the sequence $\{\zeta_n\}$ has a limit point in the disk $K: |z| < R$, then the series (8.31) must diverge, and the corollary reduces to the interior uniqueness theorem (Theorem I.16.1), specialized to the case of functions of class N. However, the interior uniqueness theorem does not apply to the case where none of the limit points of $\{\zeta_n\}$ belongs to the disk K. Thus we have found a uniqueness theorem for the class N which is stronger than the interior uniqueness theorem in the sense that it is valid under more general assumptions concerning the set E on which $f(z)$ and $g(z)$ coincide.

*38. Nevanlinna's Theorem

Next we establish an explicit characterization of the class N of functions of bounded characteristic:

THEOREM 8.7 (*Nevanlinna's theorem*). *The class N coincides with the class of all functions analytic on K which can be represented in the form*

$$f(z) = \frac{g_1(z)}{g_2(z)}, \tag{8.32}$$

where $g_1(z)$ and $g_2(z)$ are analytic and bounded on K.

Proof. Because of (8.30) and the inequality

$$\left|\ln \frac{a}{b}\right| \leqslant |\ln a| + |\ln b| \qquad (a \geqslant 0, b \geqslant 0),$$

the quotient $f(z)$ of two functions in N is also a function in N, provided $f(z)$ is analytic. In particular, any analytic function of the form (8.32), where $g_1(z)$ and $g_2(z)$ are analytic and bounded on K, belongs to N, since then $g_1(z)$ and $g_2(z)$ certainly belong to N. Thus the proof reduces to showing that any function in N has a representation of the form (8.32), where $g_1(z)$ and $g_2(z)$ have the specified properties. Suppose that $f(z) \in N$, and let $\{\rho_n\}$ be an increasing sequence of positive numbers converging to R. Denoting the circle $|z| = \rho_n$ by γ_n, for each n we use Theorem 5.8 to construct a function $u_n(z)$ harmonic on $I(\gamma_n)$ and continuous on $\overline{I(\gamma_n)}$, such that $u_n(z) = \ln^+ |f(z)|$ at every point of γ_n. According to Theorem 7.2, $u_n(z)$ is a harmonic majorant of $\ln^+ |f(z)|$. Moreover, $u_n(z)$ is nonnegative on $I(\gamma_n)$ since it takes nonnegative values on γ_n. Since $u_{n+1}(z)$ majorizes $\ln^+ |f(z)|$ on $I(\gamma_{n+1})$, we have

$$u_n(z) = \ln^+ |f(z)| \leqslant u_{n+1}(z)$$

at every point of γ_n, and hence u_{n+1} majorizes $u_n(z)$ on $I(\gamma_n)$, i.e.,

$$u_n(z) \leqslant u_{n+1}(z) \qquad (n = 1, 2, \ldots)$$

for all $z \in I(\gamma_n)$. At the origin $u_n(z)$ has the value

$$u_n(0) = \frac{1}{2\pi}\int_0^{2\pi} u_n(\rho e^{i\varphi})\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi,$$

and since $f(z) \in N$, the sequence $\{u_n(0)\}$ is bounded and hence convergent, being an increasing sequence. It follows from the corollary to Harnack's theorem (Theorem 5.12) that the sequence of harmonic functions $\{u_n(z)\}$ is uniformly convergent on every compact subset of an arbitrary disk $I(\gamma_n)$ and hence on every compact subset of K. Therefore, according to Theorem 5.11, the limit function

$$u(z) = \lim_{n \to \infty} u_n(z)$$

is harmonic and nonnegative on K, and moreover majorizes $\ln^+ |f(z)|$ on K, since

$$\ln^+ |f(z)| \leqslant u_n(z) \leqslant u(z)$$

on $I(\gamma_n)$, where n is arbitrary.

We now use Theorem 5.2 to form the harmonic conjugate $v(z)$ of the function $u(z)$. Then the function

$$g_2(z) = \exp\,[-u(z) - iv(z)]$$

is analytic and bounded by 1 on K, and has no zeros in K. The function

$$g_1(z) = f(z)g_2(z) \tag{8.33}$$

is also analytic on K, with the same zeros as $f(z)$, and moreover

$$\begin{aligned} |g_1(z)| &= |f(z)g_2(z)| \\ &= |f(z)| \exp\,[-u(z)] \leqslant \exp\,[\ln^+ |f(z)| - u(z)] \leqslant 1. \end{aligned}$$

Therefore $g_1(z)$ is also bounded on K. The proof is now completed by comparing (8.33) and (8.32).

Remark 1. Of course, the class N contains unbounded functions. For example, let K be the unit disk $|z| < 1$, and consider the function

$$f(z) = \exp\left(\frac{k}{1 - z}\right) \qquad (k > 0),$$

which becomes infinite as $z = x > 0$ approaches 1. If we write $f(z)$ in the form

$$f(z) = \frac{1}{\exp\left(\dfrac{k}{z - 1}\right)} = \frac{1}{g_2(z)},$$

then

$$|g_2(\rho e^{i\varphi})| = \left|\exp\left(\frac{k}{\rho e^{i\varphi} - 1}\right)\right| = \exp\left(\operatorname{Re}\frac{k}{\rho e^{i\varphi} - 1}\right)$$
$$= \exp\left(k\,\frac{\rho\cos\varphi - 1}{1 + \rho^2 - 2\rho\cos\varphi}\right) < 1,$$

since

$$\frac{\rho\cos\varphi - 1}{1 + \rho^2 - 2\rho\cos\varphi} < 0$$

if $\rho < 1$. Thus $f(z)$ is the quotient of two bounded analytic functions on K [here $g_1(z) \equiv 1$], i.e., $f(z) \in N$. This example shows that N contains functions for which the maximum modulus function $M(\rho)$ grows like

$$\exp\left(\frac{k}{R - \rho}\right) \qquad (k > 0)$$

as $\rho \to R$.[11]

Remark 2. Given any function $f(z) \in N$, we can find a constant $k > 0$ such that

$$M(\rho) = \max_{|z| = \rho} |f(z)| \leqslant e^{k/(R-\rho)}. \tag{8.34}$$

In fact, let $u(z)$ be the harmonic majorant of $\ln^+ |f(z)|$ found in the proof of Theorem 8.7. Then $u(z)$ is *a fortiori* a harmonic majorant of $\ln |f(z)|$. Therefore

$$\ln |f(\rho e^{i\varphi})| \leqslant u(\rho' e^{i\varphi'}) \leqslant \frac{\rho' + \rho}{\rho' - \rho}\,u(0),$$

if $\rho < \rho' < R$ (see Problem 5.18). Taking the limit as $\rho' \to R$, we find that

$$\ln |f(\rho e^{i\varphi})| \leqslant \frac{R + \rho}{R - \rho}\,u(0) < \frac{2Ru(0)}{R - r},$$

and hence

$$\ln M(\rho) = \max_{0 \leqslant \varphi \leqslant 2\pi} \ln |f(\rho e^{i\varphi})| < \frac{2Ru(0)}{R - \rho}.$$

In other words,

$$M(\rho) < e^{k/(R-\rho)},$$

where $k = 2Ru(0)$. Thus, for example, given any $\varepsilon > 0$, the function

$$f(z) = \exp\left[\frac{1}{(R - z)^{1-\varepsilon}}\right]$$

cannot belong to N, since in this case

$$M(\rho) = \exp\left[\frac{1}{(R - \rho)^{1-\varepsilon}}\right].$$

---

[11] In the present case, $R = 1$ and

$$M(\rho) = \max_{|z| = \rho} \left|\exp\left(\frac{k}{1 - z}\right)\right| = \exp\left(\frac{k}{1 - \rho}\right).$$

Finally, we note that (8.34) is a necessary but *not a sufficient* condition for the function $f(z)$ to be a member of the class N. In fact, as we know from p. 238, the function

$$f(z) = \exp\left(\frac{\pi i}{z-1}\right)$$

does not belong to N, despite the fact that it satisfies the inequality

$$M(\rho) \leqslant e^{\pi/(1-\rho)}.$$

PROBLEMS

8.1. Verify that the improper integral (8.8) is absolutely convergent.

Hint. Writing $\zeta_0 = \rho e^{i\varphi_0}$, note that

$$|\rho e^{i\varphi} - \rho e^{i\varphi_0}| = 2\rho \sin\frac{|\varphi - \varphi_0|}{2} \geqslant \frac{2}{\pi}\rho|\varphi - \varphi_0|,$$

and hence

$$|\ln|\rho e^{i\varphi} - \zeta_0|\,| \leqslant \left|\ln\left(\frac{2}{\pi}\rho|\varphi - \varphi_0|\right)\right|$$

if $|\rho e^{i\varphi} - \rho e^{i\varphi_0}| < 1$.

8.2. Evaluate the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|\rho e^{i\varphi} - a|\, d\varphi \qquad (a \neq 0)$$

as a function of $\rho \geqslant 0$.

8.3. Let $f(z)$ be analytic on the disk $|z| < R$. Under what conditions is

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(\rho e^{i\varphi})|\, d\varphi$$

a *strictly* increasing function of ρ $(0 \leqslant \rho < R)$?

8.4. Prove the following generalization of Theorem 8.3: If $h(z)$ is subharmonic on the disk $|z| < R$, then

$$\frac{1}{2\pi}\int_0^{2\pi} h(\rho e^{i\varphi})\, d\varphi$$

is an increasing function of ρ $(0 \leqslant \rho < R)$.

Hint. Use Theorems 5.8 and 7.2.

8.5. Let $\{\zeta_k\}$ be an increasing sequence of points in the disk $|z| < R$ such that

$$\sum_{k=1}^{\infty} (R - |\zeta_k|) < \infty$$

(cf. Theorem 8.5). Show that

$$\lim_{\rho \to R} \nu(\rho) \ln\frac{\rho}{R} = 0,$$

where $v(\rho)$ is the number of points of the sequence $\{\zeta_k\}$ which belong to the disk $|z| < \rho$ $(\rho < R)$.

Hint. Cf. Problem 10.2.

8.6. Prove that the Blaschke product $b(z)$ given by (8.25) has zero logarithmic mean in the sense that

$$\lim_{\rho \to R} \frac{1}{2\pi} \int_0^{2\pi} \ln |b(\rho e^{i\varphi})| \, d\varphi = 0. \tag{8.35}$$

Hint. Use formula (8.11) and the result of Problem 8.5.

8.7. According to Theorem 8.5 and Problem 8.6, the Blaschke product $b(z)$ given by (8.25) is bounded by 1 on the disk $K: |z| < R$ and satisfies the condition (8.35). Conversely, let $f(z)$ be a function which is analytic and bounded by 1 on K and satisfies (8.35). Show that

$$f(z) = e^{i\alpha} b(z),$$

where α is real and

$$b(z) = \lim_{n \to \infty} b_n(z)$$

is the Blaschke product formed from the zeros of $f(z)$ by using (8.23).

Hint. Show that $f(z) = g(z)b(z)$, where $g(z)$ is analytic and bounded by 1 on K and has no zeros in K.

8.8. Prove that if $h(z)$ is superharmonic on a domain G (see Problem 7.1), then so is the function $h^-(z)$ defined on p. 236.

8.9. Let $f(z)$ be analytic on the disk $|z| < R$. Prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})| \, d\varphi$$

is an increasing function of ρ $(0 < \rho < R)$.

Comment. It follows that $\overline{\lim}$ can be replaced by lim in (8.29).

8.10. Let $f(z)$ be a function of class N (on the disk $K: |z| < R$), so that, in particular, $f(z)$ has a unique representation

$$f(z) = F(z)b(z),$$

of the form (8.26). Prove that

$$\begin{aligned} \overline{\lim_{\rho \to R}} \frac{1}{2\pi} \int_0^{2\pi} |\ln |f(\rho e^{i\varphi})|| \, d\varphi &= \overline{\lim_{\rho \to R}} \frac{1}{2\pi} \int_0^{2\pi} |\ln |F(\rho e^{i\varphi})|| \, d\varphi \\ &= \lim_{\rho \to R} \frac{1}{2\pi} \int_0^{2\pi} 2 \ln^+ |F(\rho e^{i\varphi})| \, d\varphi - \ln |F(0)|. \end{aligned}$$

Hint. Use formula (8.35) and note that $\ln |F(\rho e^{i\varphi})|$ is harmonic on K.

8.11. Suppose $f(z)$ is analytic on the disk $K: |z| < R$ and has a representation of the form

$$f(z) = \frac{g_1(z)}{g_2(z)},$$

where $g_1(z)$ and $g_2(z)$ are analytic and bounded on K. Then $g_2(z)$ has a unique representation

$$g_2(z) = G_2(z)b_2(z)$$

of the form (8.26). Prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(\rho e^{i\varphi})|\, d\varphi \leqslant \ln \frac{1}{|G_2(0)|}.$$

Hint. Use the fact that

$$\ln \frac{1}{|G_2(z)|}$$

is harmonic on K.

Comment. This gives another proof of the fact that $f(z) \in N$ (besides Theorem 8.7).

Part 3

ENTIRE AND MEROMORPHIC FUNCTIONS

9

BASIC PROPERTIES OF ENTIRE FUNCTIONS

39. Growth of an Entire Function

It will be recalled that by an *entire function* (synonymously, an *integral function*) is meant a single-valued function which is analytic everywhere in the finite plane. Such a function has a Taylor series expansion

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots, \tag{9.1}$$

which converges for all finite z. Moreover, according to the Cauchy-Hadamard formula (I.16.2),

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = 0. \tag{9.2}$$

There are three possible ways in which an entire function $f(z)$ can behave at infinity:

1. $f(z)$ can have a regular point at infinity, and then, according to Liouville's theorem (Theorem I.16.8), $f(z) \equiv \text{const}$;
2. $f(z)$ can have a pole of order $k \geqslant 1$ at infinity, and then $f(z)$ reduces to a polynomial;
3. $f(z)$ can have an essential singular point at infinity, and then $f(z)$ is said to be an *entire transcendental function.*[1]

[1] We have already discussed polynomials in Secs. I.35–I.37, and the particularly simple entire transcendental functions e^z, $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ in Secs. I.38–I.42.

From now on, we shall be concerned predominantly with entire transcendental functions. If $f(z)$ is such a function, the most important characteristic of $f(z)$ is the *maximum modulus (function)*

$$M(r) = \max_{|z| = r} |f(z)|.$$

Clearly $M(r)$ is a strictly increasing function of r, such that

$$\lim_{r \to \infty} M(r) = \infty$$

(this follows from the maximum modulus principle and Liouville's theorem).

THEOREM 9.1. *If $f(z)$ is an entire transcendental function, with maximum modulus function $M(r)$, then*

$$\lim_{r \to \infty} \frac{\ln M(r)}{\ln r} = \infty.$$

Proof. Suppose the contrary, i.e., suppose that

$$\lim_{r \to \infty} \frac{\ln M(r)}{\ln r} = \mu < \infty.$$

Then, given any finite $\mu' > \mu$, we can find an increasing sequence $\{r_n\}$ converging to infinity such that

$$\ln M(r_n) < \mu' \ln r_n$$

or

$$M(r_n) < r_n^{\mu'}$$

for every r_n. Therefore, using Cauchy's inequalities [cf. (I.16.43)], we obtain the following estimate of the coefficients of the series (9.1):

$$|a_k| \leqslant \frac{M(r_n)}{r_n^k} < r_n^{\mu' - k}.$$

Since r_n can be chosen arbitrarily large, it follows that $a_k = 0$ for all $k > \mu' > \mu$, and hence $f(z)$ is a polynomial of degree no higher than $[\mu]$.[2] This contradiction proves the theorem.

Qualitatively speaking, Theorem 9.1 asserts that an entire transcendental function $f(z)$ grows faster than any fixed positive power of r. This suggests using the exponential function (the "simplest rapidly growing function") to measure the growth of $f(z)$.[3] Thus suppose there exists a positive number μ such that

$$M(r) < e^{r\mu} \tag{9.3}$$

[2] By $[\mu]$ is meant the *integral part* of μ, i.e., the largest integer $\leqslant \mu$.

[3] More precisely, the growth of $|f(z)|$ or of $M(r)$.

for all sufficiently large r. Then the entire function $f(z)$ is said to be *of finite order*, and the greatest lower bound

$$\rho = \inf \mu \geqslant 0 \tag{9.4}$$

of the values of μ for which (9.3) holds, starting from some sufficiently large $r = r(\mu)$, is called the *order* of $f(z)$. However, suppose the quantity (9.4) is infinite, i.e., suppose that given any positive number μ, there exist arbitrarily large values of r such that $M(r)$ exceeds e^{r^μ}. Then $f(z)$ is said to be *of infinite order*. For example, e^z is of finite order (in fact, of order 1), while e^{e^z} is of infinite order.

THEOREM 9.2. *The order ρ of an entire function is given by the formula*

$$\rho = \overline{\lim_{r \to \infty}} \frac{\ln \ln M(r)}{\ln r}. \tag{9.5}$$

Proof. According to (9.3) and (9.4), given any $\varepsilon > 0$, there is a number $R(\varepsilon) > 0$ such that

$$M(r) < e^{r^{\rho + \varepsilon}}$$

for all $r > R(\varepsilon)$, while, on the other hand, there is a sequence $\{r_n\}$, where $0 < r_1 < \cdots < r_n < \cdots$, such that

$$M(r_n) > e^{r_n^{\rho - \varepsilon}}.$$

In other words,

$$\frac{\ln \ln M(r)}{\ln r} < \rho + \varepsilon$$

if $r > R(\varepsilon)$, while

$$\frac{\ln \ln M(r_n)}{\ln r_n} > \rho - \varepsilon$$

for arbitrarily large r_n. But this is precisely what is meant by (9.5).

Next we subdivide the class of entire functions of finite order ρ. Given a function $f(z)$ of order ρ, suppose there exists a positive number K such that

$$M(r) < e^{Kr^\rho} \tag{9.6}$$

for all sufficiently large r. Then $f(z)$ is said to be *of finite type*, and the greatest lower bound

$$\sigma = \inf K \geqslant 0 \tag{9.7}$$

of the values of K for which (9.6) holds, starting from some sufficiently large $r = r(K)$, is called the *type* of $f(z)$. For example, e^z is of order $\rho = 1$ and type $\sigma = 1$. Moreover, $f(z)$ is said to be *of normal type* if $\sigma > 0$, and *of minimum type* if $\sigma = 0$. However, suppose the quantity (9.7) is infinite, i.e., suppose that given any positive number K, there exist arbitrarily large values of r such that $M(r)$ exceeds e^{Kr^ρ}. Then $f(z)$ is said to be *of infinite* (or *maximum*) *type*.

THEOREM 9.3. *The type σ of an entire function of order ρ is given by the formula*

$$\sigma = \overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r^\rho}. \tag{9.8}$$

Proof. According to (9.6) and (9.7), given any $\varepsilon > 0$, there is a number $R(\varepsilon) > 0$ such that

$$M(r) < e^{(\sigma+\varepsilon)r^\rho}$$

for all $r > R(\varepsilon)$, while, on the other hand, there is a sequence $\{r_n\}$, where $0 < r_1 < \cdots < r_n < \cdots$, such that

$$M(r_n) > e^{(\sigma-\varepsilon)r^\rho}.$$

In other words,

$$\frac{\ln M(r)}{r^\rho} < \sigma + \varepsilon$$

if $r > R(\varepsilon)$, while

$$\frac{\ln M(r_n)}{r_n^\rho} > \sigma - \varepsilon$$

for arbitrarily large r_n. But this is precisely what is meant by (9.8).

Remark. Obviously, we could have used (9.5) and (9.8) to *define* the order and type of an entire function, instead of (9.4) and (9.7).

Example. Consider the function

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Since

$$|\sinh y| \leqslant |\sin z| = \sqrt{\sinh^2 y + \sin^2 x} \leqslant \cosh y$$

[cf. p. I.148], we have

$$\frac{e^r - 1}{2} < \frac{e^r - e^{-r}}{2} = \sinh r \leqslant M(r) = \max_{|z|=r} |\sin z|$$
$$\leqslant \cosh r = \frac{e^r + e^{-r}}{2} < \frac{e^r + 1}{2},$$

i.e., $\rho = 1$ and $\sigma = 1$. The function $\cos z$ is also of order 1 and type 1.

40. Behavior of $e^{P(z)}$

The maximum modulus function $M(r)$ describes the growth of an entire function $f(z)$ in the whole complex plane Π, but gives no information at all

about the behavior of $f(z)$ in various unbounded subdomains of Π. For example, consider the exponential function e^z (of order 1 and type 1), with modulus

$$|e^z| = e^x = e^{r\cos\theta} \qquad (z = re^{i\theta}).$$

In every closed angle of the form[4]

$$-\frac{\pi}{2} + \varepsilon \leqslant \theta \leqslant \frac{\pi}{2} + \varepsilon \qquad (\varepsilon > 0),$$

we have

$$e^{r\sin\varepsilon} \leqslant |e^z| \leqslant e^r,$$

and hence $e^z \to \infty$ as $r \to \infty$. However, in every closed angle of the form

$$\frac{\pi}{2} + \varepsilon \leqslant \theta \leqslant \frac{3\pi}{2} - \varepsilon \qquad (\varepsilon > 0),$$

we have

$$|e^z| \leqslant e^{-r\sin\varepsilon},$$

and hence $e^z \to 0$ as $r \to \infty$. Thus, in this case, there are two open angles, each of π radians, i.e., the right half-plane G_0: Re $z > 0$ and the left half-plane G_1: Re $z < 0$, such that

$$\lim_{r\to\infty} e^z = \infty$$

in every closed angle contained in G_0, while

$$\lim_{r\to\infty} e^z = 0$$

in every closed angle contained in G_1.

More generally, consider the function

$$f(z) = e^{P(z)},$$

where $P(z)$ is a polynomial of degree n:

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n \qquad (a_n \neq 0, n \geqslant 1).$$

Writing

$$a_k = \sigma_k e^{i\alpha_k}, \qquad z = re^{i\theta},$$

we have

$$|f(z)| = \left|\exp\left[\sum_{k=0}^{n} \sigma_k r^k e^{i(\alpha_k + k\theta)}\right]\right| = \exp\left[\sum_{k=0}^{n} \sigma_k r^k \cos(\alpha_k + k\theta)\right]$$
$$= \exp\left\{\sigma_n r^n \left[\cos(\alpha_n + n\theta) + \sum_{k=0}^{n-1} \frac{\sigma_k \cos(\alpha_k + k\theta)}{\sigma_n r^{n-k}}\right]\right\}.$$

---

[4] By an *angle A* (with its vertex at the origin) we mean the geometric figure consisting of two rays $\theta = \alpha$ and $\theta = \beta$ ($\theta = \arg z$, $\alpha < \beta$), called the *sides* of the angle. To keep things simple, the open set $\alpha < \theta < \beta$ will be called an *open angle* (previously called the interior of *A*), and its closure $\alpha \leqslant \theta \leqslant \beta$ will be called a *closed angle*. Thus the adjectives *open* and *closed* serve to distinguish "angular regions" from their boundaries.

The sum

$$\sum_{k=0}^{n-1} \frac{\sigma_k \cos(\alpha_k + k\theta)}{\sigma_n r^{n-k}}$$

approaches zero as $r \to \infty$, and hence, given any $\varepsilon > 0$, the quantity in (square) brackets can be made less than $1 + \varepsilon$ for all sufficiently large r, say $r > R(\varepsilon)$. In other words,

$$|f(z)| < e^{(1+\varepsilon)\sigma_n r^n} \quad \text{if} \quad r > R(\varepsilon).$$

On the other hand, since $\cos(\alpha_n + n\theta) = 1$ for $\theta = -\alpha_n/n$, the quantity in brackets exceeds $1 - \varepsilon$ for the same values of r, i.e.,

$$|f(re^{-i\alpha_n/n})| > e^{(1-\varepsilon)\sigma_n r^n} \qquad \text{if} \quad r > R(\varepsilon).$$

Together, the last two inequalities imply

$$e^{(1-\varepsilon)\sigma_n r^n} < M(r) < e^{(1+\varepsilon)\sigma_n r^n} \qquad \text{if} \quad r > R(\varepsilon).$$

It follows that $f(z)$ is a function of integral order n and type σ_n.

We now divide the plane into $2n$ equal open angles

$$G_j\colon -\frac{\alpha_n}{n} + (2j-1)\frac{\pi}{2n} < \theta < -\frac{\alpha_n}{n} + (2j+1)\frac{\pi}{2n} \qquad (j = 0, 1, \ldots, 2n-1),$$

with their common vertex at the origin. We also introduce $2n$ closed angles

$$\bar{D}_j\colon -\frac{\alpha_n}{n} + (2j-1)\frac{\pi}{2n} + \frac{\varepsilon}{n} \leqslant \theta \leqslant -\frac{\alpha_n}{n} + (2j+1)\frac{\pi}{2n} - \frac{\varepsilon}{n}$$
$$(j = 0, 1, \ldots, 2n-1),$$

where $0 < \varepsilon < \pi/2$ and $\bar{D}_j \subset G_j$. At every point of $\bar{D}_j$ we have

$$j\pi - \frac{\pi}{2} + \varepsilon \leqslant \alpha_n + n\theta \leqslant j\pi + \frac{\pi}{2} - \varepsilon,$$

and hence

$$|\cos(\alpha_n + n\theta)| \geqslant \sin\varepsilon,$$

where

$$\cos(\alpha_n + n\theta) \geqslant \sin\varepsilon$$

if j is even (0 is regarded as an even number), and

$$\cos(\alpha_n + n\theta) \leqslant -\sin\varepsilon$$

if j is odd. Moreover

$$|f(z)| = \exp\left\{\sigma_n \cos(\alpha_n + n\theta) r^n \left[1 + \sum_{k=0}^{n-1} \frac{\sigma_k \cos(\alpha_k + k\theta)}{\sigma_n \cos(\alpha_n + n\theta) r^{n-k}}\right]\right\}$$

if $z \in \bar{D}_j$, and the quantity

$$\left|\sum_{k=0}^{n-1} \frac{\sigma_k \cos(\alpha_k + k\theta)}{\sigma_n \cos(\alpha_n + n\theta) r^{n-k}}\right| \leqslant \frac{1}{\sin\varepsilon} \sum_{k=0}^{n-1} \frac{\sigma_k}{\sigma_n r^{n-k}}$$

can be made less than ε if r exceeds some positive number $R(\varepsilon)$. Therefore

$$|f(z)| > \exp[\sigma_n r^n (1-\varepsilon)\sin\varepsilon]$$

if $r > R(\varepsilon)$ and $z \in \bar{D}_j$ where j is even, while

$$|f(z)| < \exp\left[-\sigma_n r^n(1 - \varepsilon)\sin\varepsilon\right]$$

if $r > R(\varepsilon)$ and $z \in \bar{D}_j$ where j is odd. It follows that

$$\lim_{\substack{|z|=r\to\infty \\ z\in\bar{D}_j}} f(z) = \begin{cases} \infty & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

The situation is shown schematically in Figure 9.1, for the case of the function e^{iz^2}.

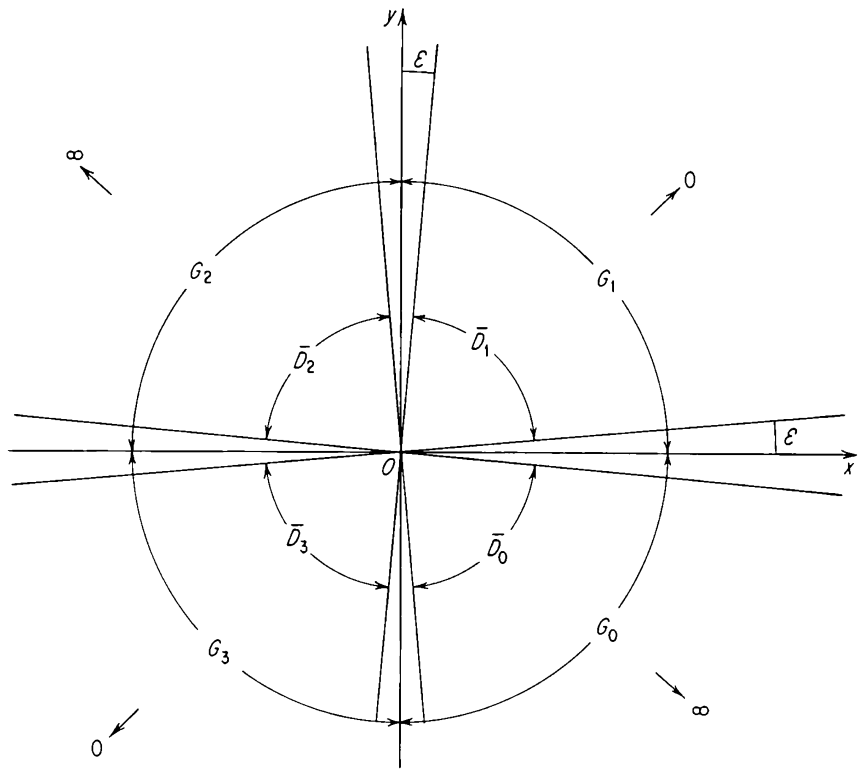


FIGURE 9.1

41. Order and Type in Terms of the Taylor Coefficients

In this section we show how the order and type of an entire function depend on the coefficients of its Taylor series expansion. We begin by proving the following

LEMMA. *Suppose $f(z)$ has the Taylor series expansion* (9.1), *and suppose there exist numbers $\mu > 0, K > 0$ and an integer $N = N(\mu, K) > 0$*

such that

$$|a_n| < \left(\frac{e\mu K}{n}\right)^{n/\mu} \tag{9.9}$$

for all $n > N$. *Then* $f(z)$ *is entire, and, given any* $\varepsilon > 0$, *there is a number* $R = R(\varepsilon) > 0$ *such that*

$$M(r) < e^{(K+\varepsilon)r^\mu} \tag{9.10}$$

for all $r > R$, *where* $M(r)$ *is the maximum modulus function of* $f(z)$.

Proof. It follows from (9.9) that

$$\sqrt[n]{|a_n|} < \left(\frac{e\mu K}{n}\right)^{1/\mu} \tag{9.11}$$

for all $n > N$, so that, in particular, $\sqrt[n]{|a_n|} \to 0$ as $n \to \infty$, i.e., $f(z)$ is entire [cf. (9.2)]. Writing (9.11) in the form

$$\sqrt[n]{|a_n|r^n} < \left(\frac{e\mu K}{n}\right)^{1/\mu} r,$$

we note that the right-hand side is less than $\frac{1}{2}$ if $n > n_0 = n_0(r)$, where

$$n_0(r) = [2^\mu e\mu K r^\mu].$$

Choosing $R' = R'(\mu, K) > 1$ so large that $n_0 = n_0(r) > N = N(\mu, K)$ if $r > R'$, we can assert that

$$\sqrt[n]{|a_n|r^n} < \frac{1}{2} \quad \text{or} \quad |a_n|r^n < \frac{1}{2^n} \tag{9.12}$$

if $r > R'$ and $n > n_0$.

We now use (9.12) to deduce an upper bound for $M(r)$. Since

$$M(r) = \max_{|z|=r} |f(z)| = \max_{|z|=r} \sum_{n=0}^{\infty} a_n z^n \leqslant \sum_{n=0}^{\infty} |a_n|r^n$$

$$= \sum_{n=0}^{n_0} |a_n|r^n + \sum_{n=n_0+1}^{\infty} |a_n|r^n,$$

we have

$$M(r) \leqslant \sum_{n=0}^{n_0} |a_n|r^n + \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} < \sum_{n=0}^{n_0} |a_n|r^n + 1$$

if $r > R'$. But

$$\sum_{n=0}^{n_0} |a_n|r^n = \sum_{n=0}^{N} |a_n|r^n + \sum_{n=N+1}^{n_0} |a_n|r^n$$

$$< r^N \sum_{n=0}^{N} |a_n| + (n_0 - N) \max_{N+1 \leqslant n \leqslant n_0} |a_n|r^n,$$

and moreover

$$\max_{N+1\leqslant n\leqslant n_0}|a_n|r^n \leqslant \max_{N+1\leqslant n}|a_n|r^n$$
$$< \max_{N+1\leqslant n}\left(\frac{e\mu K}{n}\right)^{n/\mu} r^n \leqslant \max_{1\leqslant n}\left(\frac{e\mu K}{n}\right)^{n/\mu} r^n, \tag{9.13}$$

where we use the inequality (9.9), valid for $n \geqslant N+1$. It is easy to see that the last quantity on the right in (9.13) equals e^{Kr^μ}, where the maximum is achieved for $n = \mu K r^\mu$ (here we regard n as a continuous variable).[5] Therefore

$$\max_{N+1\leqslant n\leqslant n_0}|a_n|r^n \leqslant e^{Kr^\mu},$$

and hence, if $r > R'$,

$$M(r) < r^N \sum_{n=0}^{N}|a_n| + (n_0 - N)e^{Kr^\mu} < r^N \sum_{n=0}^{N}|a_n| + (2^\mu e\mu K r^\mu - N)e^{Kr^\mu}$$
$$= e^{Kr^\mu}\left[2^\mu e\mu K r^\mu - N + r^N e^{-Kr^\mu}\sum_{n=0}^{N}|a_n|\right]. \tag{9.14}$$

Obviously, given any $\varepsilon > 0$, there is a number $R = R(\varepsilon) > R'$ such that the expression in brackets is less than $e^{\varepsilon r^\mu}$, provided that $r > R$. But then (9.14) becomes the inequality (9.10), and the lemma is proved.

THEOREM 9.4. *If $f(z)$ is an entire function of order ρ, then*[6]

$$\rho = \overline{\lim_{n\to\infty}}\frac{\ln n}{\ln\dfrac{1}{\sqrt[n]{|a_n|}}}. \tag{9.15}$$

Proof. Suppose ρ is finite (the case of infinite ρ will be considered later). Then, given any $\mu > \rho$, there is a number $R = R(\mu) > 0$ such that

$$M(r) < e^{r^\mu}$$

for all $r > R$. Therefore, according to Cauchy's inequalities,

$$|a_n| \leqslant \frac{M(r)}{r^n} < \frac{e^{r^\mu}}{r^n}$$

for all $r > R$. It is easy to see that the last quantity on the right achieves its minimum value, equal to $(e\mu/n)^{n/\mu}$, when $r = (n/\mu)^{1/\mu}$.

---

[5] Set the logarithmic derivative of

$$\left(\frac{e\mu K}{n}\right)^{n/\mu} r^n$$

equal to zero.

[6] The quotient in (9.15) is taken to be zero if $a_n = 0$.

Choosing an integer $N = N(\mu) > 0$ such that $(n/\mu)^{1/\mu} > R(\mu)$ if $n > N$, we find that

$$|a_n| < \left(\frac{e\mu}{n}\right)^{n/\mu} \tag{9.16}$$

if $n > N$, i.e.,

$$\mu > \frac{\ln n}{\ln \frac{1}{\sqrt[n]{|a_n|}}} + \frac{\ln \frac{1}{e\mu}}{\ln \frac{1}{\sqrt[n]{|a_n|}}}$$

and hence

$$\mu \geqslant \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln \frac{1}{\sqrt[n]{|a_n|}}}. \tag{9.17}$$

In the last step, we have used the fact that

$$\overline{\lim_{n \to \infty}} \ln \frac{1}{\sqrt[n]{|a_n|}} = \infty,$$

which follows from (9.2). Since μ is an arbitrary number exceeding ρ, (9.17) implies the inequality

$$\rho \geqslant \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln \frac{1}{\sqrt[n]{|a_n|}}}, \tag{9.18}$$

where the right-hand side is obviously finite.

Now let μ' be any number exceeding the right-hand side of (9.15). Then there is an integer $N = N(\mu') > 0$, not to be confused with $N(\mu)$, such that

$$|a_n| < \left(\frac{1}{n}\right)^{n/\mu'}$$

for all $n > N(\mu')$. Applying the lemma, with $\mu = \mu'$ and $e\mu' K = 1$, we find that given any $\varepsilon > 0$, there is a number $R = R(\varepsilon) > 0$, not to be confused with $R(\mu)$, such that

$$M(r) < \exp\left[\left(\frac{1}{e\mu'} + \varepsilon\right) r^{\mu'}\right]$$

for all $r > R$. Therefore $\rho \leqslant \mu'$, and hence

$$\rho \leqslant \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln \frac{1}{\sqrt[n]{|a_n|}}}, \tag{9.19}$$

because of the choice of μ'. Comparing (9.18) and (9.19), we obtain (9.15). Moreover, if the right-hand side of (9.19) is finite, so is ρ. In other words, if ρ is infinite, so is the right-hand side of (9.15), and the proof is complete.

COROLLARY. *If the right-hand side of* (9.15) *is finite and equal to* ρ, *then* $f(z)$ *is an entire function of order* ρ. *If* $f(z)$ *is entire and if the right-hand side of* (9.15) *is infinite, then* $f(z)$ *is of infinite order.*

THEOREM 9.5. *If* $f(z)$ *is an entire function of finite order* ρ *and type* σ, *then*

$$\sigma = \frac{1}{e\rho} \overline{\lim_{n \to \infty}} \, n|a_n|^{\rho/n}. \tag{9.20}$$

Proof. Suppose σ is finite (the case of infinite σ will be considered later). Then, given any $K > \sigma$, there is a number $R = R(K) > 0$ such that

$$M(r) < e^{Kr^\rho}$$

for all $r > R$. Therefore, according to Cauchy's inequalities,

$$|a_n| \leqslant \frac{M(r)}{r^n} < \frac{e^{Kr^\rho}}{r^n}$$

for all $r > R$. By the same reasoning which leads to (9.16), we find that

$$|a_n| < \left(\frac{e\rho K}{n}\right)^{n/\rho} \tag{9.21}$$

if $n > N$, where $N = N(K) > 0$ is such that $(n/\rho K)^{1/\rho} > R(K)$ if $n > N$. Writing (9.21) in the form

$$K > \frac{1}{e\rho} n|a_n|^{\rho/n},$$

we see that

$$K \geqslant \frac{1}{e\rho} \overline{\lim_{n \to \infty}} \, n|a_n|^{\rho/n}. \tag{9.22}$$

Since K is an arbitrary integer exceeding σ, (9.22) implies the inequality

$$\sigma \geqslant \frac{1}{e\rho} \overline{\lim_{n \to \infty}} \, n|a_n|^{\rho/n}, \tag{9.23}$$

where the right-hand side is obviously finite.

Now let K' be any number exceeding the right-hand side of (9.20). Then there is an integer $N = N(K') > 0$, not to be confused with $N(K)$, such that

$$|a_n| < \left(\frac{e\rho K'}{n}\right)^{n/\rho}$$

for all $n > N(K')$. Applying the lemma, with $K = K'$ and $\mu = \rho$, we find that given any $\varepsilon > 0$, there is a number $R = R(\varepsilon) > 0$, not to be confused with $R(K)$, such that

$$M(r) < e^{(K' + \varepsilon)r^\rho}$$

for all $r > R$. Therefore $\sigma \leqslant K'$, and hence

$$\sigma \leqslant \frac{1}{e\rho} \overline{\lim_{n \to \infty}} \, n|a_n|^{\rho/n}, \tag{9.24}$$

because of the choice of K'. Comparing (9.23) and (9.24), we obtain (9.20). Moreover, if the right-hand side of (9.24) is finite, so is σ. In other words, if σ is infinite, so is the right-hand side of (9.20), and the proof is complete.

COROLLARY. *If the right-hand side of* (9.20) *is finite and equal to* σ, *then* $f(z)$ *is either an entire function of order* ρ *and type* σ, *or an entire function of order less than* ρ. *If* $f(z)$ *is an entire function of finite order* ρ *and if the right-hand side of* (9.20) *is infinite, then* $f(z)$ *is of maximum type.*

Example 1. A particularly simple class of entire functions, called *functions of exponential type*, is characterized by the condition

$$\overline{\lim_{n \to \infty}} \, n\sqrt[n]{|a_n|} < \infty.$$

According to Theorem 9.5 and its corollary, these functions are either of order 1 and finite type, or of order $\leqslant 1$.

Example 2. According to Theorems 9.4 and 9.5 and their corollaries, the function

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{e\rho\sigma}{n}\right)^{n/\rho} z^n \qquad (\rho > 0, \sigma > 0)$$

is of order ρ and type σ.

Example 3. The condition

$$\lim_{n \to \infty} \frac{\ln n}{\ln \dfrac{1}{\sqrt[n]{|a_n|}}} = 0$$

characterizes an entire function of order zero. Thus any function with coefficients such that

$$|a_n| = \frac{1}{n^{n/\varepsilon_n}},$$

where $\{\varepsilon_n\}$ is an arbitrary sequence of positive numbers converging to zero, e.g., the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{n^{1+\delta}}} \qquad (\delta < 0),$$

is of order zero.

Example 4. The condition

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\ln \frac{1}{\sqrt[n]{|a_n|}}} = \infty,$$

together with (9.2), characterizes an entire function of infinite order. For example, these conditions are satisfied if

$$|a_n| = \frac{1}{n^{n\varepsilon_n}},$$

where $\{\varepsilon_n\}$ is an arbitrary sequence of positive numbers which converges to zero slowly enough so that

$$\lim_{n\to\infty} \varepsilon_n \ln n = \infty.$$

The sequence

$$\varepsilon_n = \frac{1}{(\ln n)^{1-\delta}} \qquad (n = 1, 2, \ldots)$$

meets these requirements if $0 < \delta < 1$. Therefore the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{e^{n\delta \ln n}} \qquad (0 < \delta < 1)$$

represents an entire function of infinite order.

42. Distribution of Zeros

Next we examine the relation between the distribution of zeros of an entire function and its growth:

THEOREM 9.6. *Let $n(r)$ be the number of zeros of an entire function $f(z)$ in the closed disk $|z| \leqslant r$, not including possible zeros at the origin. Then, given any $\varepsilon > 0$,*

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r^{\rho+\varepsilon}} \leqslant e(\rho + \varepsilon) \tag{9.25}$$

if $f(z)$ is of finite order $\rho \geqslant 0$, while

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r^{\rho}} \leqslant e\rho\sigma \tag{9.26}$$

if $f(z)$ is of finite order $\rho > 0$ and finite type $\sigma \geqslant 0$.

Proof. According to Jensen's inequality (8.13),

$$\int_0^r \frac{n(t)}{t}\, dt \leqslant \ln \frac{M(r)}{\left|\frac{f^{(\lambda)}(0)}{\lambda!}\right| r^{\lambda}}, \tag{9.27}$$

where $M(r)$ is the maximum modulus function of $f(z)$, and λ is the order of its zero at the origin (λ may equal zero). Let δ be an arbitrary number in the interval (0, 1). Since

$$\int_0^r \frac{n(t)}{t}\,dt \geqslant \int_{\delta r}^r \frac{n(t)}{t}\,dt \geqslant n(\delta r)\int_{\delta r}^r \frac{dt}{t} = n(\delta r)\ln\frac{1}{\delta},$$

it follows from (9.27) that

$$n(\delta r) \leqslant \frac{1}{\ln\dfrac{1}{\delta}}\ln\frac{M(r)}{\left|\dfrac{f^{(\lambda)}(0)}{\lambda!}\right| r^{\lambda}}. \tag{9.28}$$

We can write (9.28) in the form

$$\frac{n(\delta r)}{\ln M(r)} \leqslant \frac{1}{\ln\dfrac{1}{\delta}} - \frac{\ln\left|\dfrac{f^{(\lambda)}(0)}{\lambda!}\right| r^{\lambda}}{\ln M(r)\ln\dfrac{1}{\delta}},$$

valid for all sufficiently large r. Letting r approach infinity, and noting that

$$\lim_{r\to\infty}\frac{\ln r}{\ln M(r)} = 0$$

(cf. Theorem 9.1), we find that

$$\overline{\lim_{r\to\infty}}\,\frac{n(\delta r)}{\ln M(r)} \leqslant \frac{1}{\ln\dfrac{1}{\delta}}. \tag{9.29}$$

Roughly speaking, (9.29) means that the number of zeros of an entire function in the disk $|z| \leqslant \delta r$ is asymptotically no larger than $\ln M(r)$ times a certain factor depending only on the choice of δ.

Next suppose there exist two positive numbers K and μ such that

$$\ln M(r) < Kr^{\mu}$$

for all sufficiently large r. Then (9.29) implies

$$\overline{\lim_{r\to\infty}}\,\frac{n(\delta r)}{Kr^{\mu}} \leqslant \frac{1}{\ln\dfrac{1}{\delta}}$$

or

$$\overline{\lim_{r\to\infty}}\,\frac{n(r)}{r^{\mu}} \leqslant \frac{K}{\delta^{\mu}\ln\dfrac{1}{\delta}}, \tag{9.30}$$

where we have replaced δr by r. Suppose we choose δ, until now an arbitrary number in the interval (0, 1), in such a way as to minimize

the right-hand side of (9.30). It is easily seen that this minimum values is achieved for $\delta = e^{-1/\mu}$. With this choice of δ, (9.30) becomes

$$\overline{\lim_{r\to\infty}} \frac{n(r)}{r^{\mu}} \leqslant e\mu K. \tag{9.31}$$

The proof is completed by noting that if $f(z)$ is of finite order $\rho \geqslant 0$, we can set $\mu = \rho + \varepsilon$ and $K = 1$ in (9.31), obtaining (9.25), while if $\rho > 0$ and $f(z)$ is of finite type $\sigma \geqslant 0$, we can set $\mu = \rho$ and $K = \sigma + \varepsilon$ in (9.31), and then pass to the limit $\varepsilon \to 0$, obtaining (9.26).[7]

COROLLARY. *Let $f(z)$ be an entire function with infinitely many zeros, given by the increasing sequence*[8]

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, a_1, \ldots, a_n, \ldots.$$

Then, given any $\varepsilon > 0$,

$$\overline{\lim_{n\to\infty}} \frac{n}{|a_n|^{\rho+\varepsilon}} \leqslant e(\rho + \varepsilon) \tag{9.32}$$

if $f(z)$ is of finite order $\rho \geqslant 0$, while

$$\overline{\lim_{n\to\infty}} \frac{n}{|a_n|^{\rho}} \leqslant e\rho\sigma \tag{9.33}$$

if $f(z)$ is of finite order $\rho > 0$ and finite type $\sigma \geqslant 0$.

Proof. If $|a_n| = r$, then $n(r) \geqslant n$. Therefore the left-hand sides of the inequalities (9.25) and (9.26) can only be made smaller if we replace $n(r)$ by n and $|a_n|$ by r, thereby obtaining (9.32) and (9.33).

43. A-Points of Entire Functions

We now establish a number of results involving A-points of entire functions:

THEOREM 9.7. *If an entire function $f(z)$ has no A_0-points ($A_0 \neq \infty$) in the finite plane, then $f(z)$ is of the form*

$$f(z) = A_0 + e^{g(z)},$$

where $g(z)$ is an entire function.

---

[7] Note that one cannot pass to the limit $\varepsilon \to 0$ in (9.25). The inequalities (9.25) and (9.26) are trivial if $f(z)$ has only finitely many zeros.

[8] More explicitly, $|a_n| \leqslant |a_{n+1}|$ ($n = 1, 2, \ldots$). Note that the sequence $\{a_n\}$ necessarily converges to infinity (why?).

Proof. Since $f(z) - A_0$ does not vanish, the function

$$h(z) = \frac{f'(z)}{f(z) - A_0} = \frac{d}{dz} \operatorname{Ln} [f(z) - A_0]$$

is entire. Therefore

$$\int_0^z h(\zeta)\, d\zeta = \operatorname{Ln} [f(\zeta) - A_0] \Big|_{\zeta=0}^{\zeta=z},$$

where the integral is along any path joining the points 0 and z. It follows that

$$f(z) - A_0 = \exp\left\{\int_0^z h(\zeta)\, d\zeta + \ln [f(0) - A_0]\right\} = e^{g(z)},$$

where

$$g(z) = \int_0^z h(\zeta)\, d\zeta + \ln [f(0) - A_0]$$

is an entire function.

Next we prove two lemmas which are of interest in their own right:

LEMMA 1. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a function which is analytic in a neighborhood $|z| < R (R < \infty)$ *of the origin, and suppose the inequality*

$$\operatorname{Re} f(z) = u(r, \theta) \leqslant U$$

holds for all r *in the interval* $(0, R)$.[9] *Then the Taylor series coefficients satisfy the inequality*

$$|a_n| \leqslant \frac{2(U - \alpha_0)}{R^n} \qquad (n = 1, 2, \ldots), \tag{9.34}$$

where

$$\alpha_0 = \operatorname{Re} a_0 = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)\, d\theta.$$

Proof. If we write

$$f(z) = u(r, \theta) + iv(r, \theta),$$

then the function

$$U - f(z) = U - u(r, \theta) - iv(r, \theta)$$

has the Taylor series

$$U - a_0 - \sum_{n=1}^{\infty} a_n z^n,$$

[9] Note that we do not take the absolute value of $u(r, \theta)$, and hence U may be negative.

and hence

$$-a_n = \frac{1}{2\pi i}\int_{|z|=r}\frac{U - f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi r^n}\int_0^{2\pi}[U - u(r,\theta) - iv(r,\theta)]e^{-in\theta}\,d\theta \qquad (n = 1, 2, \ldots). \quad (9.35)$$

Moreover, according to Cauchy's integral theorem,

$$0 = \frac{1}{2\pi i}\int_{|z|=r}[U - f(z)]z^{n-1}\,dz = \frac{r^n}{2\pi}\int_0^{2\pi}[U - u(r,\theta) - iv(r,\theta)]e^{in\theta}\,d\theta \qquad (n = 1, 2, \ldots),$$

which implies

$$0 = \frac{1}{2\pi r^n}\int_0^{2\pi}[U - u(r,\theta) + iv(r,\theta)]e^{-in\theta}\,d\theta. \quad (9.36)$$

Adding (9.35) and (9.36), we obtain

$$-a_n = \frac{1}{\pi r^n}\int_0^{2\pi}[U - u(r,\theta)]e^{-in\theta}\,d\theta.$$

It follows that

$$|a_n| \leqslant \frac{1}{\pi r^n}\int_0^{2\pi}[U - u(r,\theta)]\,d\theta = \frac{2U}{r^n} - \frac{1}{\pi r^n}\int_0^{2\pi}u(r,\theta)\,d\theta. \quad (9.37)$$

But

$$\frac{1}{\pi r^n}\int_0^{2\pi}u(r,\theta)\,d\theta = \frac{2}{r^n}\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\,d\theta = \frac{2u(0)}{r^n} = \frac{2\alpha_0}{r^n},$$

which together with (9.37) gives

$$|a_n| \leqslant \frac{2(U - \alpha_0)}{r^n} \qquad (n = 1, 2, \ldots)$$

for all r in the interval $(0, R)$. Passing to the limit $r \to R$, we obtain the inequality (9.34).

LEMMA 2. *If the real part $u(r, \theta)$ of an entire function $f(z)$ satisfies an inequality of the form*

$$u(r, \theta) \leqslant r^\mu \qquad (\mu > 0)$$

for all sufficiently large r, then $f(z)$ is a polynomial of degree no higher then $[\mu]$.

Proof. By the maximum modulus principle for harmonic functions (cf. Theorem 5.7),

$$u(\rho, \theta) \leqslant r^\mu$$

for all $\rho < r$. Therefore, according to Lemma 1,

$$|a_n| \leqslant 2\frac{r^\mu - \alpha_0}{r^n}. \quad (9.38)$$

Choose $n = [\mu] + 1$ and let $r \to \infty$ (why is this permissible?). Then (9.38) implies

$$a_{[\mu]+1} = a_{[\mu]+2} = \cdots = 0,$$

and hence

$$f(z) = a_0 + a_1 z + \cdots + a_{[\mu]} z^{[\mu]},$$

as asserted.

THEOREM 9.8. *If an entire function $f(z)$ of order ρ has no A_0-points ($A_0 \neq \infty$) in the finite plane, then ρ is an integer and $f(z)$ is of the form*

$$f(z) = A_0 + e^{P(z)},$$

where $P(z)$ is a polynomial of degree ρ.

Proof. According to Theorem 9.7, $f(z)$ can be written in the form

$$f(z) = A_0 + e^{g(z)},$$

where $g(z)$ is an entire function. Since $f(z)$ is of order ρ, then, given any $\varepsilon > 0$, there is a number $R(\varepsilon) > 0$ such that

$$|f(z)| < e^{r^{\rho + \frac{1}{2}\varepsilon}}$$

and hence

$$|e^{g(z)}| < |A_0| + e^{r^{\rho + \frac{1}{2}\varepsilon}},$$

provided that $r > R(\varepsilon)$. Now choose $R'(\varepsilon) \geqslant R(\varepsilon)$ such that

$$|A_0| + e^{r^{\rho + \frac{1}{2}\varepsilon}} < e^{r^{\rho + \varepsilon}}$$

if $r > R'(\varepsilon)$. Then since

$$|e^{g(z)}| = e^{\operatorname{Re} g(z)},$$

we have

$$e^{\operatorname{Re} g(z)} < e^{r^{\rho + \varepsilon}}$$

if $r > R'(\varepsilon)$. Applying Lemma 2, we see that $g(z)$ is a polynomial $P(z)$.[10] But since $f(z)$ is of order ρ, the same is obviously true of the function

$$f(z) - A_0 = e^{P(z)},$$

and moreover, as shown on p. 254, $e^{P(z)}$ is of order n, where n is the degree of $P(z)$. Therefore ρ equals an integer n, and $P(z)$ is a polynomial of degree ρ, as asserted.

THEOREM 9.9. *Suppose that $z_1, \ldots, z_m$ are the distinct A_0-points ($A_0 \neq \infty$) of an entire function $f(z)$, where z_j is of order k_j ($j = 1, \ldots, m$). Then $f(z)$ is of the form*

$$f(z) = A_0 + (z - z_1)^{k_1} \cdots (z - z_m)^{k_m} e^{g(z)}, \tag{9.39}$$

where $g(z)$ is an entire function.

---

[10] In fact, a polynomial of degree no higher than $[\rho + \varepsilon]$, i.e., of degree no higher than $[\rho]$.

Proof. By hypothesis, the function

$$F(z) = \frac{f(z) - A_0}{(z - z_1)^{k_1} \cdots (z - z_m)^{k_m}}$$

is entire and has no zeros. Therefore, according to Theorem 9.7,

$$F(z) = e^{g(z)},$$

where $g(z)$ is entire. But this is just another way of writing (9.39).

THEOREM 9.10. *Suppose that in addition to satisfying the conditions of Theorem* 9.9, $f(z)$ *is known to be of order* ρ. *Then* ρ *is an integer and* $f(z)$ *is of the form*

$$f(z) = A_0 + (z - z_1)^{k_1} \cdots (z - z_m)^{k_m} e^{P(z)},$$

where $P(z)$ *is a polynomial of degree* ρ.

Proof. According to Theorem 9.9, $f(z)$ is of the form (9.39), where $g(z)$ is an entire function. Since

$$|e^{g(z)}| = e^{\operatorname{Re} g(z)} < \frac{|f(z)| + |A_0|}{|z - z_1|^{k_1} \cdots |z - z_m|^{k_m}} < |f(z)| + |A_0|,$$

provided that

$$|z - z_1| > 1, \ldots, |z - z_m| > 1,$$

and since, given any $\varepsilon > 0$,

$$|f(z)| + |A_0| < e^{r^{\rho + \varepsilon}}$$

for sufficiently large r, say $r > R(\varepsilon)$, we have

$$\operatorname{Re} g(z) < r^{\rho + \varepsilon}$$

if $r > R(\varepsilon)$. Therefore, according to Lemma 2, $g(z)$ is a polynomial $P(z)$ of degree no higher than $[\rho]$. In particular, the order of the function

$$\frac{f(z) - A_0}{(z - z_1)^{k_1} \cdots (z - z_m)^{k_m}} = e^{P(z)}$$

cannot exceed ρ. On the other hand, the order of $e^{P(z)}$ cannot be less than ρ, since this would imply

$$|e^{P(z)}| < e^{r^{\rho - 2\alpha}},$$

i.e.,

$$|f(z)| < |A_0| + e^{r^{\rho - 2\alpha}}(r + |z_1|)^{k_1} \cdots (r + |z_m|)^{k_m}, \qquad (9.40)$$

for some $\alpha > 0$ and $r > R'(\alpha)$. However, the ratio of the right-hand side of (9.40) to $e^{r^{\rho-\alpha}}$ obviously approaches zero as $r \to \infty$, and hence (9.40) reduces to

$$|f(z)| < e^{r^{\rho-\alpha}}$$

for $r > R''(\alpha) \geqslant R'(\alpha)$, contrary to the assumption that $f(z)$ is of order ρ. Therefore $e^{P(z)}$ is of order ρ, which means that ρ equals an integer n, i.e., the degree of the polynomial $P(z)$.

44. Picard's First Theorem

The culmination of the results proved in the preceding section is given by

THEOREM 9.11 (*Picard's first theorem*).[11] *Let $f(z)$ be an entire transcendental function of finite order ρ, and let A be any finite complex number. Then the set of A-points of $f(z)$ is infinite unless ρ is a positive integer and $f(z)$ is of the form*

$$f(z) = A_0 + p(z)e^{P(z)},$$

where $p(z)$ and $P(z)$ are polynomials. In this case, there are only finitely many A_0-points, which coincide with the zeros of $p(z)$, but the set of A-points is infinite if $A \neq A_0$.

Proof. Roughly speaking, the theorem asserts that there is at most one value (called *Picard's exceptional value*) for which an entire transcendental function of finite order can fail to have infinitely many A-points. If $f(z)$ has only finitely many A_0-points, of orders $k_1, \ldots, k_m$, respectively, then, according to Theorem 9.9, ρ must be an integer and $f(z)$ must be of the form

$$f(z) = A_0 + (z - z_1)^{k_1} \cdots (z - z_m)^{k_m} e^{P(z)}, \tag{9.41}$$

where $P(z)$ is a polynomial of degree ρ. The case $\rho = 0$ is excluded, since then (9.41) reduces to a polynomial, contrary to the hypothesis that $f(z)$ is an entire transcendental function (note that a polynomial has finitely many A-points for any $A \neq \infty$). Thus, if ρ is not a positive integer, $f(z)$ has infinitely many A-points for any $A \neq \infty$.

To complete the proof, we must show that if $f(z)$ has the special form (9.41), then there is no other finite number $B_0 \neq A_0$ for which

[11] Also known as *Picard's little theorem*. Note the resemblance between Theorem 9.11 and Picard's second (or great) theorem, cited on p. 21. The restriction to entire functions of finite order is not intrinsic, but is merely a limitation of the elementary method of proof chosen here. As will be shown in Volume III of this course, Theorem 9.11 is valid for entire functions of arbitrary order, finite or infinite.

$f(z)$ has finitely many B_0-points. Suppose the opposite, i.e., suppose there is such a number B_0. Then, according to Theorem 9.10, $f(z)$ can be written in the form

$$f(z) = B_0 + (z - \zeta_1)^{l_1} \cdots (z - \zeta_n)^{l_n} e^{Q(z)}, \tag{9.42}$$

where $Q(z)$ is also a polynomial of degree ρ. It follows from (9.41) and (9.42) that

$$p(z)e^{P(z)} - q(z)e^{Q(z)} \equiv C, \tag{9.43}$$

where $C = B_0 - A_0 \neq 0$, and

$$p(z) = (z - z_1)^{k_1} \cdots (z - z_m)^{k_m},$$
$$q(z) = (z - \zeta_1)^{l_1} \cdots (z - \zeta_n)^{l_n}.$$

However, the relation (9.43) is impossible, as we now show, using an argument due to D. A. Raikov.

Differentiation of (9.43) gives

$$[p'(z) + p(z)P'(z)]e^{P(z)} - [q'(z) + q(z)Q'(z)]e^{Q(z)} \equiv 0. \tag{9.44}$$

If

$$p'(z) + p(z)P'(z) \equiv 0, \tag{9.45}$$

then $p(z) \equiv$ const, since otherwise the function

$$P'(z) = -\frac{p'(z)}{p(z)}$$

would have poles at the zeros of $p(z)$, which is impossible, since $P(z)$ is a polynomial. But $p(z) \equiv$ const is also impossible, since then $P(z) \equiv$ const, contrary to the hypothesis that $P(z)$ is a polynomial of positive degree. Exactly the same argument shows that the relation

$$q'(z) + q(z)Q'(z) \equiv 0 \tag{9.46}$$

is impossible.[12] Therefore (9.44) can be written in the form

$$\frac{p'(z) + p(z)P'(z)}{q'(z) + q(z)Q'(z)} = e^{Q(z) - P(z)}.$$

The expression on the left must equal a constant, since it is a rational function with no poles or zeros in the finite plane. Therefore

$$Q(z) - P(z) \equiv \text{const},$$

and (9.43) implies

$$p(z) - C_1 q(z) = C_2 e^{-P(z)},$$

where C_1 and C_2 are nonzero constants. But the expression on the left is a polynomial with no zeros, i.e., a constant, and hence $P(z) \equiv$ const,

[12] Because of (9.44), either of the relations (9.45) and (9.46) implies the other.

contrary to hypothesis. Therefore the relation (9.43) is impossible, and the proof is complete.[13]

Example. We illustrate Theorem 9.11 by finding the roots of the equation

$$\sin z - Az = 0 \tag{9.47}$$

for arbitrary $A \neq \infty$. If $A = 0$, (9.47) reduces to the equation $\sin z = 0$, with infinitely many roots of the form $z = k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$). If $A \neq 0$, the roots of (9.47) are just the A-points of the entire function

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,$$

which is easily seen to be of order $\rho = 1$.[14] Since ρ is a positive integer, the possibility is not excluded that there exists a finite $A_0 \neq 0$ such that the set of A_0-points of $f(z)$ [and hence the set of roots of (9.47)] is finite. However, consider the entire function

$$g(\zeta) = \frac{\sin \sqrt{\zeta}}{\sqrt{\zeta}} = 1 - \frac{\zeta}{2!} + \frac{\zeta}{4!} - \cdots,$$

which is easily seen to be of order $\frac{1}{2}$. Setting $z = \sqrt{\zeta}$, we can write (9.47) in the form

$$\frac{\sin \sqrt{\zeta}}{\sqrt{\zeta}} = A. \tag{9.48}$$

Obviously, every root $\zeta \neq 0$ of (9.48) leads to two roots of (9.47), and therefore if (9.48) has infinitely many roots, so does (9.47). But the roots of (9.48) are A-points of an entire function of order $\frac{1}{2}$, and according to Theorem 9.11, the set of such A-points is infinite for any $A \neq \infty$. Therefore (9.47) has infinitely many roots for any $A \neq \infty$.

*45. The Phragmén-Lindelöf Indicator Function

Using the Phragmén-Lindelöf theorems of Sec. 34, we can deduce valuable information about the behavior of an entire function of finite order inside an angle from a knowledge of its behavior on the sides of the angle:

THEOREM 9.12. *Let G be the interior of an angle of $\alpha\pi$ radians with boundary Γ and vertex at the origin, and let $f(z)$ be an entire function whose order does not exceed ρ. Suppose that $\alpha < 1/\rho$ and*

$$|f(z)| \leqslant C < \infty$$

[13] For another way of proving that (9.43) is impossible, which is more direct but less elegant, see Problem 9.14.

[14] Express the maximum modulus function of $f(z)$ in terms of the exponential function, as in the example on p. 252, or else use formula (9.15).

for every finite point $\zeta \in \Gamma$. *Then*

$$|f(z)| \leqslant C$$

for all $z \in G$. *Moreover, this conclusion remains valid for* $\alpha = 1/\rho$ *if* $f(z)$ *is of order* ρ *and minimum type.*

Proof. The condition

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r^{1/\alpha}} = 0 \tag{9.49}$$

implies the condition

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r^{1/\alpha}} \leqslant 0$$

figuring in Theorem 7.6. But (9.49) means that $f(z)$ is either of order less than $1/\alpha$, or of order $1/\alpha$ and minimum type (i.e., type zero).

COROLLARY 1. *Given a finite family of rays emanating from the origin, suppose the angle between every pair of adjacent rays does not exceed* π/ρ, *where* $\rho \geqslant \frac{1}{2}$. *Then every entire function* $f(z) \not\equiv$ const *of order less than* ρ *is unbounded on at least one ray of the family.*

Proof. If $f(z)$ were bounded on all the rays, it would also be bounded between the rays and hence in the whole plane. But this would imply $f(z) \equiv$ const, contrary to hypothesis.

COROLLARY 2. *Let* $f(z) \not\equiv$ const *be an entire function of order less than* $\frac{1}{2}$, *or of order* $\frac{1}{2}$ *and minimum type. Then* $f(z)$ *is unbounded on every ray emanating from the origin.*

Example. Consider the function

$$f(z) = e^{P(z)},$$

where $P(z)$ is a polynomial of degree n. Then, as shown on p. 254, $f(z)$ is of order n. Suppose we draw a family of rays from the origin, such that the angle between every pair of adjacent rays is less than π/n. Then at least one of these rays must fall in each of the domains G_j ($j = 0, 1, \ldots, 2n - 1$) constructed on p. 254. In particular, at least one ray falls in each G_j with an even subscript, and $f(z)$ is unbounded on every such ray.

Remark. Theorem 9.12 is no longer valid if $\alpha = 1/\rho$ and $f(z)$ is of order ρ and type $\sigma > 0$. For example, consider the function

$$f(z) = \sin z$$

of order 1 and type 1, and let G be the upper (or lower) half-plane,[15] whose

[15] Clearly, any half-plane can be regarded as an angle of π radians.

boundary Γ is the real axis. Then $f(z)$ is bounded on Γ, but not on G itself. Similarly, Corollary 2 is no longer valid if $f(z)$ is of order $\frac{1}{2}$ and type $\sigma > 0$, as shown by the function

$$f(z) = \frac{\sin \sqrt{z}}{\sqrt{z}},$$

which approaches zero as $z \to \infty$ along the positive real axis.

Let $f(z)$ be an entire function of finite order $\rho > 0$ and finite type $\sigma \geqslant 0$. To study $f(z)$ in more detail, we introduce the function

$$h(\theta) = \overline{\lim_{r \to \infty}} \frac{\ln |f(re^{i\theta})|}{r^\rho},$$

which measures the growth of $|f(z)|$ along the ray making the angle θ with the positive real axis. The function $h(\theta)$, called the *(Phragmén-Lindelöf) indicator function*, is obviously single-valued and periodic, with period 2π. Since

$$\overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r^\rho} = \sigma < \infty,$$

we have

$$h(\theta) \leqslant \sigma, \tag{9.50}$$

i.e., $h(\theta)$ is bounded from above.

THEOREM 9.13. *If θ_1 and θ_2 satisfy the condition*

$$0 < \theta_2 - \theta_1 < \min\left(\frac{\pi}{\rho}, 2\pi\right), \tag{9.51}$$

and if a and b are finite real numbers such that

$$h(\theta_1) \leqslant a, \qquad h(\theta_2) \leqslant b, \tag{9.52}$$

then

$$h(\theta) \leqslant A \cos \rho\theta + B \sin \rho\theta \qquad (\theta_1 \leqslant \theta \leqslant \theta_2), \tag{9.53}$$

where A and B are determined by the system of equations

$$\begin{aligned} A \cos \rho\theta_1 + B \sin \rho\theta_1 &= a, \\ A \cos \rho\theta_2 + B \sin \rho\theta_2 &= b. \end{aligned} \tag{9.54}$$

Moreover, given any $\varepsilon > 0$, the inequality

$$\ln |f(re^{i\theta})| < (A \cos \rho\theta + B \sin \rho\theta + \varepsilon) r^\rho \qquad (\theta_1 \leqslant \theta \leqslant \theta_2) \tag{9.55}$$

holds for all sufficiently large $r > R(\varepsilon)$.

Proof. Given any $\eta > 0$, define $\alpha = \alpha(\eta)$ and $\beta = \beta(\eta)$ by the equations

$$\begin{aligned} \alpha \cos \rho\theta_1 + \beta \sin \rho\theta_1 &= a + \eta, \\ \alpha \cos \rho\theta_2 + \beta \sin \rho\theta_2 &= b + \eta. \end{aligned} \tag{9.56}$$

This is possible, since the determinant of the system (9.56) is $\sin \rho(\theta_2 - \theta_1) \neq 0$. It is obvious that

$$\lim_{\eta \to 0} \alpha(\eta) = A, \qquad \lim_{\eta \to 0} \beta(\eta) = B. \tag{9.57}$$

Now consider the function

$$F_\eta(z) = f(z) \exp [-(\alpha - i\beta)z^\rho],$$

which is single-valued[16] and analytic in the open angle

$$G\colon 0 < r < \infty, \qquad \theta_1 < \theta < \theta_2.$$

At every point of G, $F_\eta(z)$ satisfies the inequality

$$\begin{aligned} |F_\eta(re^{i\theta})| &= |f(z)| \exp [-(\alpha \cos \rho\theta + \beta \sin \rho\theta)r^\rho] \\ &< \exp [(\sigma + 1)r^\rho - (\alpha \cos \rho\theta + \beta \sin \rho\theta)r^\rho] < \exp (Kr^\rho) \end{aligned} \tag{9.58}$$

for some $K = K(\eta)$ and sufficiently large $r > R_1(\eta)$. Moreover, because of (9.52), we have

$$\begin{aligned} |F_\eta(re^{i\theta_1})| &< \exp [(a + \eta)r^\rho - (\alpha \cos \rho\theta_1 + \beta \sin \rho\theta_1)r^\rho] = 1, \\ |F_\eta(re^{i\theta_2})| &< \exp [(b + \eta)r^\rho - (\alpha \cos \rho\theta_2 + \beta \sin \rho\theta_2)r^\rho] = 1 \end{aligned}$$

for sufficiently large $r > R_2(\eta)$. Therefore $|F_\eta(z)|$ is bounded on the sides of G, i.e.,

$$|F_\eta(re^{i\theta_1})| \leqslant C, \qquad |F_\eta(re^{i\theta_2})| \leqslant C \qquad (0 \leqslant r < \infty), \tag{9.59}$$

where $C = C(\eta)$. It follows from (9.58) and (9.59), and from the condition (9.51), that Theorem 7.5 can be applied to the function $F_\eta(z)$ and the domain G (the interior of an angle of $\theta_2 - \theta_1$ radians). Thus

$$|F_\eta(z)| \leqslant C \qquad (z \in \bar{G}),$$

which implies

$$\ln |f(re^{i\theta})| \leqslant \ln C + (\alpha \cos \rho\theta + \beta \sin \rho\theta)r^\rho \qquad (re^{i\theta} \in \bar{G}). \tag{9.60}$$

Now, given any $\varepsilon > 0$, choose $\eta = \eta(\varepsilon)$ so small that

$$|\alpha(\eta) - A| < \frac{\varepsilon}{3}, \qquad |\beta(\eta) - B| < \frac{\varepsilon}{3}$$

[recall (9.57)], and then choose a number $R(\varepsilon) > 0$ such that

$$\left| \frac{\ln C}{r^\rho} \right| < \frac{\varepsilon}{3}$$

if $r > R(\varepsilon)$. Combining these inequalities with (9.60), we immediately obtain (9.55). But (9.55) implies

$$h(\theta) = \overline{\lim_{r \to \infty}} \frac{\ln |f(re^{i\theta})|}{r^\rho} \leqslant A \cos \rho\theta + B \sin \rho\theta + \varepsilon,$$

[16] Here we choose $z^\rho = (re^{i\theta})^\rho = r^\rho(\cos \rho\theta + i \sin \rho\theta)$, as in the proof of Theorem 7.5.

and then (9.53) follows at once, since ε is arbitrarily small. This completes the proof.

Remark. Solving (9.54) for A and B, and substituting the result into (9.53), we find that

$$h(\theta) \leqslant a \frac{\sin \rho(\theta_2 - \theta)}{\sin \rho(\theta_2 - \theta_1)} + b \frac{\sin \rho(\theta - \theta_1)}{\sin \rho(\theta_2 - \theta_1)} \qquad (\theta_1 \leqslant \theta \leqslant \theta_2). \tag{9.61}$$

In particular, on the bisector of G we have

$$\theta_2 - \theta = \theta - \theta_1 = \frac{\theta_2 - \theta_1}{2},$$

and hence

$$h\left(\frac{\theta_1 + \theta_2}{2}\right) \leqslant \frac{1}{\cos \rho\left(\dfrac{\theta_2 - \theta_1}{2}\right)} \frac{a + b}{2}. \tag{9.62}$$

THEOREM 9.14. *The indicator function is finite for all* θ:

$$h(\theta) > -\infty \qquad (0 \leqslant \theta \leqslant 2\pi).$$

Proof. Suppose the opposite. Then $h(\theta_0) = -\infty$ for some θ_0. Choosing an integer $m > \rho$, we apply the inequality (9.62) to the bisector of the open angle

$$\theta_0 < \theta < \theta_0 + \frac{\pi}{m}.$$

The result is

$$h\left(\theta_0 + \frac{\pi}{2m}\right) \leqslant \frac{1}{\cos \dfrac{\rho\pi}{2m}} \frac{a + b}{2}, \tag{9.63}$$

where a and b are any two finite real numbers satisfying the conditions

$$-\infty = h(\theta_0) \leqslant a, \qquad h\left(\theta_0 + \frac{\pi}{2m}\right) \leqslant b.$$

Holding b fixed, we pass to the limit $a \to -\infty$ in (9.63), obtaining

$$h\left(\theta_0 + \frac{\pi}{2m}\right) = -\infty.$$

Repeating the same argument, we find that in general

$$h\left(\theta_0 + k\frac{\pi}{2m}\right) = -\infty \qquad (k = 0, 1, \ldots, 2m - 1).$$

Therefore, by the definition of the indicator function $h(\theta)$, $f(z)$ must be bounded on every ray

$$\theta = \theta_0 + k\frac{\pi}{2m} \qquad (k = 0, 1, \ldots, 2m - 1).$$

But according to Theorem 9.12 (note that the angle between any two adjacent rays is $\pi/2m < \pi/\rho$), this implies that $f(z)$ is bounded in the whole plane, i.e., $f(z) \equiv \text{const}$, contrary to the hypothesis that $f(z)$ is an entire function of order $\rho > 0$. This contradiction proves the theorem.

COROLLARY. *The inequality* (9.61) *can be written in the form*

$$h(\theta) \leqslant h(\theta_1)\frac{\sin \rho(\theta_2 - \theta)}{\sin \rho(\theta_2 - \theta_1)} + h(\theta_2)\frac{\sin \rho(\theta - \theta_1)}{\sin \rho(\theta_2 - \theta_1)} \qquad (\theta_1 \leqslant \theta \leqslant \theta_2). \tag{9.64}$$

Proof. We can always choose $a = h(\theta_1)$, $b = h(\theta_2)$ in (9.61).

THEOREM 9.15. *The indicator function is continuous for all* θ.

Proof. Choosing an arbitrary θ_0 in $[0, 2\pi]$, we set $\theta_1 = \theta_0$ in (9.64) and pass to the limit $\theta \to \theta_0$, obtaining

$$\overline{\lim_{\substack{\theta \to \theta_0 \\ \theta > \theta_0}}} h(\theta) \leqslant h(\theta_0).$$

Then, setting $\theta_2 = \theta_0$ and passing to the limit $\theta \to \theta_0$, we find that

$$\overline{\lim_{\substack{\theta \to \theta_0 \\ \theta < \theta_0}}} h(\theta) \leqslant h(\theta_0).$$

The last two inequalities imply

$$\overline{\lim_{\theta \to \theta_0}} h(\theta) \leqslant h(\theta_0). \tag{9.65}$$

Next we set $\theta = \theta_0$ in (9.64) and pass first to the limit $\theta_1 \to \theta_0$ and then to the limit $\theta_2 \to \theta_0$. The result is

$$h(\theta_0) \leqslant \lim_{\substack{\theta_1 \to \theta_0 \\ \theta_1 < \theta_0}} h(\theta_1), \qquad h(\theta_0) \leqslant \lim_{\substack{\theta_2 \to \theta_0 \\ \theta_2 > \theta_0}} h(\theta_2),$$

i.e.,

$$h(\theta_0) \leqslant \lim_{\theta \to \theta_0} h(\theta). \tag{9.66}$$

Comparing (9.65) and (9.66), we obtain

$$\lim_{\theta \to \theta_0} h(\theta) = \overline{\lim_{\theta \to \theta_0}} h(\theta) = h(\theta_0),$$

as asserted.

THEOREM 9.16. *If* $\rho \geqslant \frac{1}{2}$, *there are no two numbers* θ' *and* θ'' *in* $[0, 2\pi]$ *such that*

$$\theta'' - \theta' = \frac{\pi}{\rho}$$

and

$$h(\theta') < 0, \qquad h(\theta'') < 0. \tag{9.67}$$

Proof. Suppose two such numbers exist. Then, setting

$$\theta_1 = \theta' + \varepsilon, \quad \theta_2 = \theta'' - \varepsilon, \quad a = h(\theta' + \varepsilon), \quad b = h(\theta'' - \varepsilon)$$

in (9.62), where

$$0 < \varepsilon < \frac{\theta'' - \theta'}{2},$$

we obtain

$$h\left(\frac{\theta' + \theta''}{2}\right) \leqslant \frac{h(\theta' + \varepsilon) + h(\theta'' - \varepsilon)}{2\cos\rho\left(\dfrac{\theta'' - \theta'}{2} - \varepsilon\right)} = \frac{h(\theta' + \varepsilon) + h(\theta'' - \varepsilon)}{2\sin\rho\varepsilon}.$$

As $\varepsilon \to 0$, the right-hand side approaches $-\infty$, because of (9.67) and the continuity of $h(\theta)$. Since this is impossible, the theorem is proved.

THEOREM 9.17. *If* $0 < \rho \leqslant \frac{1}{2}$, *there is no value of* θ *for which* $h(\theta) < 0$.

Proof. If $\rho = \frac{1}{2}$, the assertion is an obvious consequence of Theorem 9.16. If $0 < \rho < \frac{1}{2}$ and if $h(\theta_0) < 0$ for some θ_0, then $f(z)$ must be bounded on the corresponding ray. But according to Theorem 9.12, Corollary 2, this is impossible.

THEOREM 9.18. *If* $\rho > \frac{1}{2}$, *the indicator function satisfies the inequality*

$$-\sigma \leqslant h(\theta) \leqslant \sigma, \tag{9.68}$$

where σ *is the type of* $f(z)$.

Proof. We already know that $h(\theta) \leqslant \sigma$ [cf. (9.50)]. Suppose $h(\theta_0) < 0$ for some θ_0. Since, according to Theorem 9.16, $h(\theta)$ must be nonnegative for some values of θ, there is an open interval (θ', θ'') containing θ_0 in which $h(\theta) < 0$, while $h(\theta') = h(\theta'') = 0$. Clearly

$$0 < \theta'' - \theta' \leqslant \frac{\pi}{\rho},$$

since otherwise we could find values θ_1 and θ_2 in (θ', θ'') such that

$$\theta_2 - \theta_1 = \frac{\pi}{\rho}, \qquad h(\theta_1) < 0, \qquad h(\theta_2) < 0,$$

contrary to Theorem 9.16. Suppose, to be explicit, that θ' is the end point of (θ', θ'') which is nearest to θ_0. Then we have

$$0 < \theta_0 - \theta' = \delta \leqslant \frac{\pi}{2\rho}.$$

Setting

$$\theta_1 = \theta' - \delta + \varepsilon, \qquad \theta_2 = \theta' + \delta - \varepsilon = \theta_0 - \varepsilon \qquad (0 < \varepsilon < \delta).$$

in formula (9.64),[17] we find that

$$h(\theta') = 0 \leqslant \frac{1}{\cos \rho(\delta - \varepsilon)} \frac{h(\theta' - \delta + \varepsilon) + h(\theta' + \delta - \varepsilon)}{2},$$

which implies

$$h(\theta' - \delta + \varepsilon) \geqslant -h(\theta' + \delta - \varepsilon). \tag{9.69}$$

Since this inequality holds for arbitrary $\varepsilon > 0$ ($\varepsilon < \delta$), and since $h(\theta)$ is continuous, we can pass to the limit $\varepsilon \to 0$ in (9.69), obtaining

$$h(\theta' - \delta) \geqslant -h(\theta' + \delta) = -h(\theta_0).$$

But $h(\theta' - \delta) \leqslant \sigma$, and hence

$$h(\theta_0) \geqslant -\sigma.$$

The proof is now complete.

Remark 1. As already noted, (9.68) can be replaced by the stronger condition

$$0 \leqslant h(\theta) \leqslant \sigma \tag{9.70}$$

if $\rho \leqslant \frac{1}{2}$.

Remark 2. If $f(z)$ is of minimum type, then either (9.68) or (9.70) implies $h(\theta) \equiv 0$.

THEOREM 9.19. *The maximum value of the indicator function equals the type* σ.[18]

Proof. Suppose the theorem is false, so that

$$\max_{0 \leqslant \theta \leqslant 2\pi} h(\theta) = \sigma_1 < \sigma.$$

Then we set $a = b = \sigma_1$ in (9.54), solve for A and B, and substitute the results into the inequality (9.55), obtaining

$$\ln |f(re^{i\theta})| < \left[\frac{\cos \rho\left(\frac{\theta_2 + \theta_1}{2} - \theta\right)}{\cos \rho\left(\frac{\theta_2 - \theta_1}{2}\right)} \sigma_1 + \varepsilon \right] r^\rho \qquad (\theta_1 \leqslant \theta \leqslant \theta_2)$$

if $r > R(\varepsilon)$. Writing

$$\frac{\theta_2 + \theta_1}{2} = \theta_0, \qquad \frac{\theta_2 - \theta_1}{2} = \delta,$$

---

[17] Formula (9.64) is applicable, since

$$(\theta' + \delta - \varepsilon) - (\theta' - \delta + \varepsilon) = 2\delta - 2\varepsilon < \frac{\pi}{\rho}.$$

[18] Regardless of the value of ρ ($0 < \rho < \infty$).

we see that

$$\ln |f(re^{i\theta})| < \left[\frac{\cos(\theta_0 - 0)}{\cos \delta}\sigma_1 + \varepsilon\right]r^\rho$$

in the interval $[\theta_0 - \delta, \theta_0 + \delta]$ if $r > R(\varepsilon)$. As $\delta \to 0$ and $\varepsilon \to 0$, the expression in brackets converges (uniformly in θ_0 and θ) to the limit σ_1, and can therefore be made less than σ_2, where $\sigma_1 < \sigma_2 < \sigma$, for sufficiently small $\delta = \delta_0$ and $\varepsilon = \varepsilon_0$. It follows that

$$\ln |f(re^{i\theta})| < \sigma_2 r^\rho \tag{9.71}$$

if $\theta_0 - \delta_0 \leqslant \theta \leqslant \theta_0 + \delta_0$ and $r > R_0 = R_0(\theta_0)$. Obviously, we can choose δ_0 equal to $\pi/2m$, where m is a sufficiently large positive integer. Then the inequality (9.71) will hold in all the closed angles

$$\frac{k\pi}{m} \leqslant \theta \leqslant \frac{(k+1)\pi}{m} \qquad (k = 0, 1, \ldots, 2m - 1)$$

for sufficiently large r. But this implies

$$\ln |f(re^{i\theta})| < \sigma_2 r^\rho$$

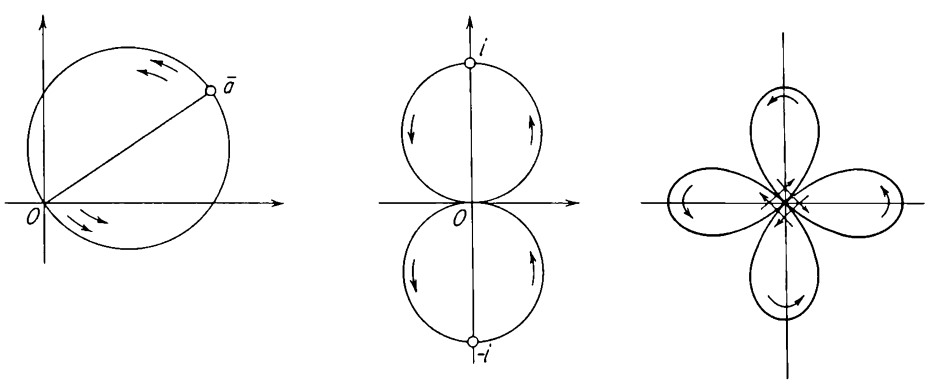
for sufficiently large r, which means that

$$\sigma = \overline{\lim_{r \to \infty}} \frac{\ln |f(re^{i\theta})|}{r^\rho} \leqslant \sigma_2.$$

This contradiction proves the theorem.

The graph in polar coordinates of the indicator function of $f(z)$, i.e., the curve

$$r = h(\theta),$$



(a) (b) (c)

FIGURE 9.2

is called the *indicator diagram* of $f(z)$. For example, consider the entire functions

a) e^{az} $(a = \alpha e^{i\varphi})$; b) $\cos z$ or $\sin z$; c) e^{z^n} (n a positive integer),

with indicator functions

a) $\alpha \cos(\theta + \varphi)$; b) $|\sin \theta|$; c) $\cos n\theta$.

The corresponding indicator diagrams are shown in Figure 9.2. The first curve is a circle which is traversed twice and has the segment joining the points 0 and $\bar{a}$ as its diameter. The second curve (corresponding to both of the functions $\cos z$ and $\sin z$) consists of two circles of unit diameter, with centers on the imaginary axis and a point of contact at the origin. The third curve is a rose with n petals if n is odd and $2n$ petals if n is even.[19]

PROBLEMS

9.1. Find the order ρ and type σ of the following entire functions:

a) z^4e^{3z}; b) $z^2e^{2z} - e^{3z}$; c) $e^{5z} - 3e^{2z^3}$; d) $e^{(2-i)z^2}$; e) $\cosh z$; f) $e^z \cos z$; g) $e^z \cosh z$; h) $\cos \sqrt{z}$.

Ans. b) $\rho = 1, \sigma = 3$; d) $\rho = 2, \sigma = \sqrt{5}$; f) $\rho = 1, \sigma = \sqrt{2}$; h) $\rho = \frac{1}{2}, \sigma = 1$.

9.2. Without recourse to Theorems 9.4 and 9.5, find the order and type of the entire function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(2^m n)!},$$

where m is a positive integer.

Hint. For example,

$$\sum_{n=0}^{\infty} \frac{z^n}{(2^2 n)!} = \frac{1}{2}(\cos \sqrt[4]{z} + \cos i\sqrt[4]{z}).$$

Ans. $\rho = 1/2^m, \sigma = 1$.

9.3. Find the order and type of the entire function

$$f(z) = \int_0^1 e^{zt^2}\, dt.$$

Ans. $\rho = 1, \sigma = 1$.

9.4. Suppose $f_1(z)$ and $f_2(z)$ are entire functions of orders ρ_1 and ρ_2, respectively, where $\rho_1 \neq \rho_2$. What can be said about the order of

a) $f_1(z) + f_2(z)$; b) $f_1(z)f_2(z)$?

[19] The case $n = 2$ is shown in Figure 9.2(c).

9.5. Suppose two entire functions $f_1(z)$ and $f_2(z)$ of the same order ρ have types σ_1 and σ_2, respectively, where $\sigma_1 \neq \sigma_2$. What can be said about the order and type of

a) $f_1(z) + f_2(z)$; b) $f_1(z)f_2(z)$?

9.6. Suppose two entire functions $f_1(z)$ and $f_2(z)$ have the same order ρ and the same type σ. What can be said about the order and type of

a) $f_1(z) + f_2(z)$; b) $f_1(z)f_2(z)$?

9.7. Given an entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

let

$$M^*(r) = \max_{n \geqslant 0} |a_n| r^n.$$

Prove that we can replace $M(r)$ by $M^*(r)$ in formula (9.5).

9.8. Without recourse to Theorems 9.4 and 9.5, prove that if $f(z)$ is of order ρ and type σ, then the same is true of the derivative $f'(z)$.

Hint. First show that

$$\frac{M(r) - |f(0)|}{r} \leqslant M_1(r) \leqslant \frac{M(r)}{R - r} \qquad (R > r),$$

where $M(r)$ and $M_1(r)$ are the maximum modulus functions of $f(z)$ and $f'(z)$, respectively.

9.9. Prove that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is an entire function of order 1 and minimum type if and only if

$$\lim_{n \to \infty} \sqrt[n]{n!\,|a_n|} = 0.$$

9.10. Solve Problems 9.2 and 9.8 with the help of Theorem 9.4 and 9.5.[20]

9.11. In connection with the corollary to Theorem 9.4, give an example of a function such that

$$\overline{\lim_{n \to \infty}} \frac{\ln n}{\ln \dfrac{1}{\sqrt[n]{|a_n|}}} = \infty,$$

but which is not entire.

9.12. Find the order ρ and type σ of the entire functions defined by the following series:

a) $\sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^n$; b) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^{a/n} z^n \qquad (a > 0)$;

c) $\sum_{n=2}^{\infty} \left(\frac{1}{n \ln n}\right)^{n/a} z^n \qquad (a > 0)$; d) $\sum_{n=0}^{\infty} e^{-n^2} z^n$.

---

[20] In Problems 9.9 and 9.10, anticipate Stirling's formula (10.108).

9.13. Let $f(z)$ be an entire function with maximum modulus $M(r)$, and suppose $|f(0)| = 1$. Prove that

$$n(r) \leqslant \ln M(er),$$

where $n(r)$ is the number of zeros of $f(z)$ in the closed disk $|z| \leqslant r$.

9.14. By using the results of Sec. 40, prove that the relation (9.43) is impossible.

Hint. Construct the open angles G_j ($j = 0, 1, \ldots, 2\rho - 1$) for the function $e^{P(z)}$, and the analogous angles G_j^* ($j = 0, 1, \ldots, 2\rho - 1$) for the function $e^{Q(z)}$. Since all the angles G_j and G_j^* are of the same size (π/ρ radians) and have a common vertex (the origin), the angle G_1, say, either contains points of some G_j^* with an even index, or else coincides with a domain G_j^* with an odd index. In the first case, there exists a closed angle $\bar{D}$ with its vertex at the origin, such that for some ε ($0 < \varepsilon < \pi/2$) and all $z \in \bar{D}$ of sufficiently large absolute value,

$$|e^{P(z)}| < \exp[-|a_\rho| r^\rho (1 - \varepsilon) \sin \varepsilon],$$

$$|e^{Q(z)}| > \exp[|b_\rho| r^\rho (1 - \varepsilon) \sin \varepsilon],$$

where a_ρ is the coefficient of z^ρ in the polynomial $P(z)$, and b_ρ is the coefficient of z^ρ in $Q(z)$. But then

$$|p(z)e^{P(z)} - q(z)e^{Q(z)}| > (|z| - |\zeta_1|)^{l_1} \cdots (|z| - |\zeta_n|)^{l_n} \exp[|b_\rho| r^\rho (1 - \varepsilon) \sin \varepsilon] - (|z| + |z_1|)^{k_1} \cdots (|z| + |z_m|)^{k_m} \exp[-|a_\rho| r^\rho (1 - \varepsilon) \sin \varepsilon],$$

where the notation is the same as on p. 269. This is incompatible with (9.43), since the right-hand side approaches ∞ as $|z| \to \infty$. In the second case, choose a closed angle $\bar{D} \subset G_1 = G_j^*$, where j is odd, and show that (9.43) again leads to a contradiction.

9.15. What is the indicator function of

a) $\cosh z$; b) $\sinh z$; c) $e^z + z^2$?

Draw the corresponding indicator diagrams.

9.16. Suppose $f(z)$ is an entire function with indicator function $h(\theta)$. What can be said about the indicator function of $f(z) + P(z)$, where $P(z)$ is a polynomial?

10

INFINITE PRODUCT AND PARTIAL FRACTION EXPANSIONS

46. Weierstrass' Theorem

Concerning the zeros of an entire transcendental function, there are only three possibilities:

1. $f(z)$ has no zeros at all;
2. $f(z)$ has a finite number of zeros;
3. $f(z)$ has an infinite number of zeros.

In the first case, according to Theorem 9.7, $f(z)$ has the representation

$$f(z) = e^{g(z)},$$

where $g(z)$ is entire. In the second case, according to Theorem 9.9,

$$f(z) = (z - z_1)^{k_1} \cdots (z - z_m)^{k_m} e^{g(z)},$$

where $g(z)$ is entire, and $z_1, \ldots, z_m$ are the distinct zeros of $f(z)$, of orders $k_1, \ldots, k_m$, respectively. We now consider the third case, where $f(z)$ has infinitely many zeros:

THEOREM 10.1 (*Weierstrass' theorem*).[1] *Given a nonnegative integer* λ *and an increasing sequence of nonzero complex numbers* $\{\zeta_n\}$ *converging to*

[1] It is instructive to compare the statement of Theorem 10.1 with that of Theorem 8.5. As always, by an increasing sequence of complex numbers we mean a sequence of complex numbers arranged in order of increasing absolute value.

infinity, there exists an entire function $f(z)$ whose zeros coincide with the points

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, \zeta_1, \ldots, \zeta_n, \ldots. \tag{10.1}$$

Proof. Consider the sequence of entire functions

$$f_m(z) = z^\lambda \prod_{n=1}^{m} \left(1 - \frac{z}{\zeta_n}\right) e^{P_n(z)} \qquad (m = 1, 2, \ldots),$$

where the $P_n(z)$ are polynomials, to be suitably chosen later. Obviously, the zeros of $f_m(z)$ coincide with the first $m + \lambda$ points of the sequence (10.1). In general, $f_m(z)$ has multiple zeros, since (10.1) can contain the same point several times (this possibility is explicitly indicated for the point $z = 0$). The idea of the proof is to choose the polynomials $P_n(z)$ in such a way that the sequence $\{f_m(z)\}$ is uniformly convergent on every compact subset, since then we can invoke Theorem I.15.8 to deduce that the limit function

$$f(z) = \lim_{m \to \infty} f_m(z) \tag{10.2}$$

is entire.

With this in mind, let K_R denote the disk $|z| < R$, and let $N(R)$ be the smallest integer such that $|\zeta_n| > 2R$ for all $n > N(R)$. Then, if $z \in K_R$ and $n > N(R)$, we can write $f_m(z)$ in the form

$$\begin{aligned} f_m(z) &= f_{N(R)}(z) \prod_{n=N(R)+1}^{m} \left(1 - \frac{z}{\zeta_n}\right) e^{P_n(z)} \\ &= f_{N(R)}(z) \exp\left\{ \sum_{n=N(R)+1}^{m} \left[\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z)\right]\right\}, \end{aligned} \tag{10.3}$$

where every logarithmic term can be expanded as a power series

$$\ln\left(1 - \frac{z}{\zeta_n}\right) = -\frac{z}{\zeta_n} - \cdots - \frac{z^n}{n\zeta_n^n} - \frac{z^{n+1}}{(n+1)\zeta_n^{n+1}} - \cdots,$$

since $|z/\zeta_n| < \frac{1}{2}$ for all $z \in K_R$ and $n > N(R)$. Choosing $P_n(z)$ so as to cancel the first n terms of this series, i.e.,

$$P_n(z) = \frac{z}{\zeta_n} + \cdots + \frac{z^n}{n\zeta_n^n}, \tag{10.4}$$

we have

$$\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z) = -\frac{z^{n+1}}{(n+1)\zeta_n^{n+1}} - \cdots, \tag{10.5}$$

which implies

$$\begin{aligned} \left|\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z)\right| &\leqslant \frac{1}{n+1}\left|\frac{z}{\zeta_n}\right|^{n+1} + \frac{1}{n+2}\left|\frac{z}{\zeta_n}\right|^{n+2} + \cdots \\ &< \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^n}. \end{aligned} \tag{10.6}$$

Then the series

$$\sum_{n=N(R)+1}^{\infty} \left[\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z)\right] \tag{10.7}$$

is uniformly convergent on K_R, since[2]

$$\sum_{n=N(R)+1}^{\infty} \left|\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z)\right| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

Therefore (10.7) represents an analytic function $\chi_R(z)$ on K_R (cf. Theorem I.15.6).

Comparing (10.2) and (10.3), and using the continuity of the exponential function, we find that

$$f(z) = f_{N(R)}e^{\chi_R(z)} \qquad (z \in K_R). \tag{10.8}$$

This shows that $f(z)$ is analytic on K_R, a fact which also follows from the uniform convergence of the sequence $\{f_m(z)\}$ on K_R (implied by Theorem I.15.15). Since the disk K_R can have arbitrarily large radius, $\{f_m(z)\}$ is uniformly convergent on every compact set, and hence the function $f(z)$ is analytic in the whole plane, i.e., $f(z)$ is entire. [This can also be seen directly from (10.8).] The fact that the zeros of $f(z)$ coincide with the points (10.1) is almost obvious, and follows from the representation (10.8) and the arbitrariness of R, since $\exp[\chi_R(z)]$ is nonvanishing, while, by construction, the zeros of $f_{N(R)}(z)$ in K_R are precisely those points of the sequence (10.1) which lie in K_R. Finally, recalling the definition of $f_m(z)$, we note that

$$f(z) = \lim_{m \to \infty} z^{\lambda} \prod_{n=1}^{m} \left(1 - \frac{z}{\zeta_n}\right) e^{P_n(z)}. \tag{10.9}$$

COROLLARY 1. *The infinite product*

$$f(z) = z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) \exp\left(\frac{z}{\zeta_n} + \cdots + \frac{z^n}{n\zeta_n}\right) \tag{10.10}$$

is an entire function satisfying the requirements of Theorem 10.1.

Proof. Equation (10.10) is another way of writing (10.9).

COROLLARY 2.[3] *Let $f(z)$ be an entire function with zeros given by the increasing sequence*

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, a_1, \ldots, a_n, \ldots.$$

---

[2] For the rest of the proof, we can substitute an argument based on the use of Theorem I.17.4.

[3] Note the analogy between this corollary and the corollary to Theorem 8.5.

Then $f(z)$ can be represented in the form

$$f(z) = e^{g(z)}z^{\lambda}\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)\exp\left(\frac{z}{a_n} + \cdots + \frac{z^n}{na_n^n}\right), \qquad (10.11)$$

where $g(z)$ is an entire function.

Proof. The function

$$\varphi(z) = z^{\lambda}\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)\exp\left(\frac{z}{a_n} + \cdots + \frac{z^n}{na_n^n}\right)$$

is entire, with the same zeros as $f(z)$, and hence the quotient $f(z)/\varphi(z)$ is entire and nonvanishing. Therefore, according to Theorem 9.7,

$$\frac{f(z)}{\varphi(z)} = e^{g(z)},$$

where $g(z)$ is entire.

47. The Exponent of Convergence

As will be shown in this section and the next, the general formula (10.11) can be greatly simplified in the case of entire functions of finite order. We begin by introducing a concept which will play a key role in these considerations.

Let $\{\zeta_n\}$ be an arbitrary increasing sequence of nonzero complex numbers which converges to infinity, and consider the series

$$\sum_{n=1}^{\infty}\frac{1}{|\zeta_n|^{\alpha}}, \qquad (10.12)$$

where α is nonnegative. If the series (10.12) converges for some $\alpha_0 > 0$, then it converges for all $\alpha > \alpha_0$ (since the numbers $|\zeta_n|^{-1}$ are all less than 1, starting from some value of n). The greatest lower bound of the values of α for which (10.12) converges is a nonnegative number τ called the *exponent of convergence* (or *convergence exponent*) of the sequence $\{\zeta_n\}$. If (10.12) diverges for all $\alpha > 0$, we set $\tau = \infty$ and say that the exponent of convergence of the sequence $\{\zeta_n\}$ is infinite.

Example. The convergence exponents of the sequences

$$\{e^n\}, \quad \{n^{1/\tau}\} \quad \text{and} \quad \{\ln(n+1)\}$$

are 0, τ and ∞, respectively.

THEOREM 10.2. *The exponent of convergence τ of the sequence $\{\zeta_n\}$ is given by the formula*

$$\tau = \overline{\lim_{n\to\infty}}\frac{\ln n}{\ln|\zeta_n|}. \qquad (10.13)$$

Proof. Suppose τ is finite (the case of infinite τ will be considered later). Then the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\alpha}}$$

converges for any $\alpha > \tau$. Since the terms of this series are nonincreasing, it follows that

$$\overline{\lim_{n \to \infty}} \frac{n}{|\zeta_n|^{\alpha}} = 0$$

(see Problem 10.2). Therefore

$$\alpha > \frac{\ln n}{\ln |\zeta_n|}$$

for all sufficiently large n, which implies

$$\alpha \geqslant \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln |\zeta_n|}$$

or

$$\tau \geqslant \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln |\zeta_n|}, \tag{10.14}$$

since α is an arbitrary number exceeding τ.

Next let α' be any number exceeding the right-hand side of (10.13). Then there is an integer $N = N(\alpha')$ such that

$$\frac{\ln n}{\ln |\zeta_n|} < \alpha'$$

for all $n > N$. Therefore

$$\frac{1}{|\zeta_n|} < n^{-1/\alpha'}$$

for all $n > N$, which means that the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\beta}}$$

converges for any $\beta > \alpha'$. It follows from the definition of the convergence exponent that $\tau \leqslant \alpha'$, and hence

$$\tau \leqslant \overline{\lim_{n \to \infty}} \frac{\ln n}{\ln |\zeta_n|}, \tag{10.15}$$

because of the definition of α'. Comparing (10.14) and (10.15), we obtain (10.13). Moreover, if the right-hand side of (10.15) is finite, so is τ. In other words, if τ is infinite, so is the right-hand side of (10.13), and the proof is complete.

COROLLARY. *If the right-hand side of* (10.13) *is finite and equal to* τ, *then the exponent of convergence of the sequence* $\{\zeta_n\}$ *equals* τ. *If the right-hand side of* (10.13) *is infinite, so is* τ.

In Theorem 10.1 we assumed no detailed information about the sequence $\{\zeta_n\}$. However, if appropriate information is available, it can be used to draw stronger conclusions:

THEOREM 10.3. *Given a nonnegative integer* λ *and an increasing sequence of nonzero complex numbers* $\{\zeta_n\}$ *converging to infinity, let* $\varkappa$ *be the largest nonnegative integer* k *for which the series*

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^k}$$

diverges.[4] *Then the expression*

$$f(z) = z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) \exp\left(\frac{z}{\zeta_n} + \cdots + \frac{z^{\varkappa}}{\varkappa \zeta_n^{\varkappa}}\right),$$

known as a canonical product, where the exponential factors disappear if $\varkappa = 0$, *represents an entire function whose zeros coincide with the points*

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, \zeta_1, \ldots, \zeta_n, \ldots.$$

Proof. The proof parallels that of Theorem 10.1, with the important difference that we now have

$$P_n(z) = \begin{cases} \dfrac{z}{\zeta_n} + \cdots + \dfrac{z^{\varkappa}}{\varkappa \zeta_n^{\varkappa}} & \text{if } \varkappa \geqslant 1, \\ 0 & \text{if } \varkappa = 0 \end{cases} \tag{10.16}$$

and

$$\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z) = -\frac{z^{\varkappa+1}}{(\varkappa+1)\zeta_n^{\varkappa+1}} - \cdots,$$

instead of (10.4) and (10.5), and

$$\left|\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z)\right| = \left|-\frac{z^{\varkappa+1}}{\zeta_n^{\varkappa+1}} \sum_{p=1}^{\infty} \frac{z^{p-1}}{(\varkappa+p)\zeta_n^{p-1}}\right| < \frac{R^{\varkappa+1}}{|\zeta_n|^{\varkappa+1}} \sum_{p=1}^{\infty} \frac{1}{2^{p-1}} = \frac{2R^{\varkappa+1}}{|\zeta_n|^{\varkappa+1}},$$

instead of (10.6). Therefore (10.7) is again uniformly convergent on $K_R: |z| < R$, but this time because of the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\varkappa+1}}$$

---

[4] It follows from the definition of $\varkappa$ that $\varkappa \leqslant [\tau]$ and $\tau \leqslant \varkappa + 1$, where τ is the convergence exponent of the sequence $\{\zeta_n\}$, and that

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\varkappa}} = \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\varkappa+1}} < \infty.$$

instead of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}.$$

The rest of the proof is identical with that of Theorem 10.1.

COROLLARY. *Let $f(z)$ be an entire function with zeros given by the increasing sequence*

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, a_1, \ldots, a_n, \ldots, \tag{10.17}$$

and let $\varkappa$ be the largest nonnegative integer k for which the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^k} \tag{10.18}$$

diverges. Then $f(z)$ can be represented in the form

$$f(z) = e^{g(z)} z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{z^{\varkappa}}{\varkappa a_n^{\varkappa}}\right), \tag{10.19}$$

where $g(z)$ is an entire function and the exponential factors disappear if $\varkappa = 0$.

Remark. Obviously, Theorem 10.3 and its corollary remain valid if we replace $\varkappa$ by any larger integer.

Next we prove an important inequality due to Hadamard:

THEOREM 10.4. *If $f(z)$ is an entire function of finite order ρ, with zeros given by* (10.17), *and if the sequence $\{a_n\}$ has convergence exponent τ, then $\tau \leqslant \rho$.*

Proof. Given any $\varepsilon > 0$, it follows from (9.32) that

$$\frac{1}{|a_n|^{\rho+\varepsilon}} < \frac{e(\rho + 2\varepsilon)}{n}$$

or

$$\frac{1}{|a_n|^{\rho+2\varepsilon}} < [e(\rho + 2\varepsilon)]^{(\rho+2\varepsilon)/(\rho+\varepsilon)} \frac{1}{n^{(\rho+2\varepsilon)/(\rho+\varepsilon)}}$$

for all sufficiently large n. Therefore the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\alpha}}$$

converges for all $\alpha \geqslant \rho + 2\varepsilon$, and hence for all $\alpha > \rho$, since $\varepsilon > 0$ is arbitrary, i.e., $\tau \leqslant \rho$, by the definition of the convergence exponent.

Remark 1. In particular, if $\varkappa$ is the largest nonnegative integer k for which the series (10.18) diverges, then $\varkappa \leqslant [\rho]$.

Remark 2. The great advantage of formula (10.19) over the general formula (10.11) is that polynomials of the same fixed degree $\varkappa$ can be used in all the exponential factors, instead of polynomials whose degree becomes arbitrarily large (with n). In the special case where $\varkappa = 0$, the exponential factors in (10.19) disappear, and we have the particularly simple infinite product expansion

$$f(z) = e^{g(z)} z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right).$$

For example, $\varkappa = 0$ if $\tau = 1$ and the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|}$$

converges. If $f(z)$ is of finite order ρ, then $\varkappa = 0$ if $\rho < 1$, or if $\rho \geqslant 1$ but $\tau < 1$.

48. Hadamard's Factorization Theorem

We now show that even more can be said about the form of the infinite product expansion of $f(z)$, if $f(z)$ is of finite order:

THEOREM 10.5 (*Hadamard's factorization theorem*). *Let $f(z)$ be an entire function of finite order ρ, with zeros given by the increasing sequence*

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, a_1, \ldots, a_n, \ldots,$$

and let $\varkappa \leqslant [\rho]$ be the largest nonnegative integer k for which the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^k}$$

diverges. Then $f(z)$ can be represented in the form (10.19), *where $g(z)$ is a polynomial whose degree does not exceed $[\rho]$, and the exponential factors disappear if $\varkappa = 0$.*

Proof. The theorem is a recapitulation of previous results, except for the assertion that $g(z)$ is a polynomial of degree no higher than $[\rho]$. Thus we can start from the expansion (10.19) if we (temporarily) assume that it is only known that $g(z)$ is an entire function. As usual, let $n(r)$ be the number of zeros in the disk $|z| \leqslant r$, not including possible zeros at the origin, and let $M(r)$ be the maximum modulus function of $f(z)$. Then, given any $R > 1$, we can write (10.19) in the form

$$f(z) = z^{\lambda} \prod_{m=1}^{n(R)} \left(1 - \frac{z}{a_m}\right) \left[e^{g(z) + P_1(z) + \cdots + P_{n(R)}(z)} \prod_{m=n(R)+1}^{\infty} \left(1 - \frac{z}{a_m}\right) e^{P_m(z)} \right],$$

where the polynomials $P_1(z), P_2(z), \ldots$ are the same as in (10.16), with $\zeta_1, \zeta_2, \ldots$ replaced by $a_1, a_2, \ldots$. Let the expression in brackets be denoted by $g_R(z)$. Then if $|z| = 2R$, we have

$$M(2R) \geqslant |f(z)| \geqslant (2R)^{\lambda}\left(\frac{2R}{R} - 1\right)^{n(R)} |g_R(z)| \geqslant |g_R(z)|,$$

and hence

$$\max_{|z|=2R} |g_R(z)| \leqslant M(2R).$$

Since $g_R(z)$ is an entire function, it follows from the maximum modulus principle that

$$|g_R(z)| < M(2R) \tag{10.20}$$

for all z in the disk $K_R: |z| < R$.

Next we observe that $g_R(z)$ has the representation

$$g_R(z) = e^{h_R(z)}$$

in terms of the function

$$h_R(z) = g(z) + \sum_{m=1}^{n(R)} P_m(z) + \sum_{m=n(R)+1}^{\infty} \left[\ln\left(1 - \frac{z}{a_m}\right) + P_m(z)\right], \tag{10.21}$$

where the second series on the right is uniformly convergent on K_R, since just as in the proof of Theorem 10.3, for sufficiently large $m > N(R)$, its terms are dominated by those of the convergent series

$$\sum_{m=1}^{\infty} \frac{2R^{\varkappa+1}}{|a_m|^{\varkappa+1}}.$$

It follows from the inequality (10.20) that

$$\operatorname{Re} h_R(z) < \ln M(2R)$$

for all $z \in K_R$. Therefore, according to Lemma 1, p. 264, the Taylor coefficients c_n of the function $h_R(z)$ satisfy the inequality

$$|c_n| = \left|\frac{h_R^{(n)}(0)}{n!}\right| < 2\,\frac{\ln M(2R) - \alpha_0}{R^n} \qquad (n = 1, 2, \ldots),$$

where

$$\alpha_0 = \operatorname{Re} c_0 = \operatorname{Re} h_R(0).$$

But the numbers c_n can be obtained by adding the coefficients of z^n in the Taylor series of the separate terms in the right-hand side of (10.21).[5] Choosing $n > \rho \geqslant \tau \geqslant \varkappa$ and recalling that $P_m(z)$ is a polynomial of degree $\varkappa$, we find that

$$c_n = \frac{g^{(n)}(0)}{n!} - \sum_{m=n(R)+1}^{\infty} \frac{1}{na_m^n},$$

[5] See Theorem I.18.1 and the subsequent remark.

and hence, if $n > \rho$,

$$\left|\frac{g^{(n)}(0)}{n!}\right| \leqslant |c_n| + \sum_{m=n(R)+1}^{\infty} \frac{1}{|a_m|^n} \leqslant 2\,\frac{\ln M(2R) - \alpha_0}{R^n} + \sum_{m=n(R)+1}^{\infty} \frac{1}{|a_m|^n}. \tag{10.22}$$

To complete the proof, let R approach infinity (for fixed n). Then the last sum on the right in (10.22) approaches zero. In fact, for $\rho + \varepsilon < n$ and sufficiently large R, we have

$$M(2R) < e^{(2R)^{\rho+\varepsilon}},$$

i.e.,

$$\ln M(2R) < (2R)^{\rho+\varepsilon},$$

and hence

$$\lim_{R\to\infty} \frac{2[\ln M(2R) - \alpha_0]}{R^n} = 0,$$

while the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_m|^n}$$

converges (since n is larger than the convergence exponent τ), and hence

$$\lim_{R\to\infty} \sum_{m=n(R)+1}^{\infty} \frac{1}{|a_m|^n} = 0.$$

It follows that

$$\frac{g^{(n)}(0)}{n!} = 0$$

if $n > \rho$, which implies that $g(z)$ is a polynomial whose degree does not exceed $[\rho]$, as asserted.

Example. The zeros of the entire function $\sin z$ are all simple, and are given by the sequence

$$0, -\pi, \pi, \ldots, -n\pi, n\pi, \ldots,$$

whose convergence exponent τ obviously equals 1. Moreover, the series

$$\frac{1}{\pi} + \frac{1}{\pi} + \cdots + \frac{1}{n\pi} + \frac{1}{n\pi} + \cdots$$

diverges, while the series

$$\frac{1}{\pi^2} + \frac{1}{\pi^2} + \cdots + \frac{1}{n^2\pi^2} + \frac{1}{n^2\pi^2} + \cdots$$

converges. Therefore the number $\varkappa$ figuring in Theorem 10.5 equals 1, and the infinite product expansion of $\sin z$ takes the form

$$\begin{aligned}\sin z &= e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n\pi}\right) e^{z/n\pi} \left(1 + \frac{z}{n\pi}\right) e^{-z/n\pi} \\ &= e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).\end{aligned}$$

Since the order of $\sin z$ equals 1, it follows from Theorem 10.5 that $g(z)$ is a polynomial of degree no higher than 1, i.e.,

$$g(z) = c_0 + c_1 z.$$

The easiest way of determining the coefficients c_0 and c_1 is to note that

$$e^{g(z)} = \frac{\sin z}{z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)} \tag{10.23}$$

is an even function, and hence

$$e^{c_0 + c_1 z} \equiv e^{c_0 - c_1 z},$$

which implies

$$e^{2c_1 z} \equiv 1 \quad \text{or} \quad c_1 = 0.$$

Moreover, passing to the limit $z \to 0$ in (10.23), we find that

$$e^{c_0} = 1,$$

which implies $c_0 = 0$ (or more generally, $c_0 = 2m\pi i$, where m is an integer). Thus finally

$$e^{g(z)} \equiv 1,$$

and hence

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right). \tag{10.24}$$

This expansion has already been derived by another, less general method on p. 63.

49. Borel's Theorem

Next we prove a result which is essentially the converse of Hadamard's factorization theorem:

THEOREM 10.6 (*Borel's theorem*). *Given a nonnegative integer* λ *and an increasing sequence of nonzero complex numbers* $\{\zeta_n\}$ *converging to infinity, let* $\tau \neq \infty$ *be the exponent of convergence of the sequence* $\{\zeta_n\}$ *and let* $\varkappa$ *be the largest nonnegative integer* k *for which the series*

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^k}$$

diverges. Then the infinite product

$$f(z) = e^{Q(z)} z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) \exp\left(\frac{z}{\zeta_n} + \cdots + \frac{z^{\varkappa}}{\varkappa \zeta_n^{\varkappa}}\right), \tag{10.25}$$

where $Q(z)$ *is a polynomial of degree* ν *and the exponential factors disappear if* $\varkappa = 0$, *represents an entire function of finite order*

$$\rho = \max(\nu, \tau), \tag{10.26}$$

whose zeros coincide with the points

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, \zeta_1, \ldots, \zeta_n, \ldots.$$

Moreover, $f(z)$ is of finite type if $\nu > \tau$, or if the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\tau}} \tag{10.27}$$

converges.

Proof. We begin by deriving an upper bound for the factor

$$\varphi\left(\frac{z}{\zeta_n}\right) = \left(1 - \frac{z}{\zeta_n}\right) \exp\left(\frac{z}{\zeta_n} + \cdots + \frac{z^{\varkappa}}{\varkappa \zeta_n^{\varkappa}}\right)$$

or

$$\varphi(\omega) = (1 - \omega) \exp\left(\omega + \cdots + \frac{\omega^{\varkappa}}{\varkappa}\right),$$

where we have written $\omega = z/\zeta_n$ to keep the notation simple. If $|\omega| \leqslant \frac{1}{2}$, then

$$\begin{aligned} |\varphi(\omega)| &\leqslant \left|\exp\left[\ln(1 - \omega) + \omega + \cdots + \frac{\omega^{\varkappa}}{\varkappa}\right]\right| \\ &= \left|\exp\left[-\frac{\omega^{\varkappa+1}}{\varkappa + 1} - \frac{\omega^{\varkappa+2}}{\varkappa + 2} - \cdots\right]\right| \\ &\leqslant \left[\exp |\omega|^{\varkappa+1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\right] = e^{2|\omega|^{\varkappa+1}} \leqslant e^{|2\omega|^{\mu'}}, \end{aligned}$$

where μ' is an arbitrary real number which does not exceed $\varkappa + 1$, while if $|\omega| > \frac{1}{2}$, then

$$\begin{aligned} |\varphi(\omega)| &< (1 + |\omega|) \exp(|\omega| + \cdots + |\omega|^{\varkappa}) \\ &\leqslant (1 + |\omega|) \exp\left[|\omega|^{\varkappa}\left(1 + \frac{1}{|\omega|} + \cdots + \frac{1}{|\omega|^{\varkappa-1}}\right)\right] \\ &< (1 + |\omega|) \exp[|\omega|^{\varkappa}(1 + 2 + \cdots + 2^{\varkappa-1})] \\ &< \exp[|2\omega|^{\varkappa} + \ln(1 + |\omega|)] \leqslant \exp[C_0|2\omega|^{\mu''}], \end{aligned}$$

where μ'' is an arbitrary positive number which is not less than $\varkappa$, and $C_0 = C_0(\mu'')$. It follows that

$$|\varphi(\omega)| \leqslant e^{C|\omega|^{\mu}} \tag{10.28}$$

for every $\mu > 0$ such that $\varkappa \leqslant \mu \leqslant \varkappa + 1$, where $C = 2^{\mu}C_0(\mu)$.

We now choose a value of μ such that the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\mu}}$$

converges. In fact, if the series (10.27) converges, then $\varkappa < \tau \leqslant \varkappa + 1$ and we set $\mu = \tau$, while if (10.27) diverges, then $\varkappa \leqslant \tau < \varkappa + 1$ and we

set $\mu = \tau + \varepsilon$, where $0 < \varepsilon < \varkappa + 1 - \tau$. Replacing ω by z/ζ_n in (10.28), we obtain the following inequality for this choice of μ and arbitrary z and n:

$$\left|\left(1 - \frac{z}{\zeta_n}\right) \exp\left(\frac{z}{\zeta_n} + \cdots + \frac{z^\varkappa}{\varkappa \zeta_n^\varkappa}\right)\right| \leqslant \exp\left(C \frac{|z|^\mu}{|\zeta_n|^\mu}\right).$$

Moreover, if

$$Q(z) = A_0 + A_1 z + \cdots + A_\nu z^\nu,$$

then, given any $\varepsilon > 0$, we have

$$|z^\lambda e^{Q(z)}| \leqslant |z|^\lambda \exp\left(|A_0| + \cdots + |A_\nu|\,|z|^\nu\right) < \exp\left[(|A_\nu| + \varepsilon)|z|^\nu\right]$$

for sufficiently large $|z| > r(\varepsilon)$, and hence

$$|f(z)| < \exp\left[(|A_\nu| + \varepsilon)|z|^\nu + C \sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^\mu} |z|^\mu\right] \tag{10.29}$$

if $|z| > R(\varepsilon)$. It follows that $f(z)$ is of finite order ρ, where

$$\rho \leqslant \max(\nu, \mu).$$

But μ either equals τ or can be chosen arbitrarily close to τ, and therefore

$$\rho \leqslant \max(\nu, \tau). \tag{10.30}$$

On the other hand, according to Theorems 10.4 and 10.5, we have

$$\tau \leqslant \rho, \qquad \nu \leqslant [\rho] \leqslant \rho,$$

which implies

$$\rho \geqslant \max(\nu, \tau). \tag{10.31}$$

Comparing (10.30) and (10.31), we obtain (10.26).

To complete the proof, we note that in the case where $\nu > \tau$ (so that $\rho = \nu$), we can choose a value of μ such that $\tau < \mu < \nu$, and then (10.29) gives

$$|f(z)| < \exp(C_1 |z|^\rho)$$

for all sufficiently large $|z|$, i.e., $f(z)$ is of finite type. Moreover, in the case where $\nu \leqslant \tau$ (so that $\rho = \tau$) and the series (10.27) converges, we can choose $\mu = \tau$ in (10.29), obtaining

$$|f(z)| < \exp(C_2 |z|^\rho),$$

and $f(z)$ is again of finite type.

Remark. It follows from Theorems 10.5 and 10.6 that the class of entire functions of finite order coincides with the class of infinite product expansions of the form (10.25).

The larger of the two integers ν and $\varkappa$, i.e., the number

$$p = \max(\nu, \varkappa),$$

is called the *genus* of $f(z)$. This concept plays an important role in certain problems involving entire functions. Clearly

$$p \leqslant \rho,$$

since

$$\varkappa \leqslant \tau, \qquad \rho = \max(\nu, \tau).$$

Moreover

$$\tau \leqslant \varkappa + 1,$$

and hence

$$\rho \leqslant \max(\nu, \varkappa) + 1 = p + 1.$$

Thus

$$p \leqslant \rho \leqslant p + 1,$$

corresponding to the two possibilities

$$p = [\rho] \quad \text{or} \quad p = [\rho] - 1.$$

It is easy to see that the second of these formulas holds if and only if τ is an integer, $\nu < \tau$ and the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\tau}}$$

converges.

Example. For the function

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

[cf. (10.24)], we have $\nu = 0$, $\tau = \varkappa = 1$ and hence $\rho = \max(\nu, \tau) = 1$, $p = \max(\nu, \varkappa) = 1 = \rho$, while for the function

$$f(z) = \prod_{n=2}^{\infty} \left(1 - \frac{z}{n(\ln n)^2}\right),$$

we have $\nu = 0$, $\tau = 1$, $\varkappa = 0$ and hence $\rho = 1$, $p = 0$. Thus two functions can have the same order and different genus.

Next we return to the subject of A-points of entire functions:

THEOREM 10.7. *The convergence exponent of the sequence of A-points of an entire function $f(z)$ of fractional order ρ is equal to ρ for every A.*

Proof. The functions $f(z)$ and $f(z) - A$ are both of order ρ, with expansions

$$f(z) = e^{g(z)} z^{\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{z^{\varkappa}}{\varkappa a_n^{\varkappa}}\right),$$

$$f(z) - A = e^{h(z)} z^{\lambda_A} \prod_{n=1}^{\infty} \left(1 - \frac{z}{b_n}\right) \exp\left(\frac{z}{b_n} + \cdots + \frac{z^{\mu}}{\mu a_n^{\mu}}\right),$$

where

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, a_1, \ldots, a_n, \ldots$$

are the zeros of $f(z)$, and

$$\underbrace{0, \ldots, 0}_{\lambda_A \text{ times}}, b_1, \ldots, b_n, \ldots$$

are the zeros of $f(z) - A$, i.e., the A-points of $f(z)$. Suppose the polynomials $g(z)$ and $h(z)$ have degrees ν and ν_A, while the sequences $\{a_n\}$ and $\{b_n\}$ have convergence exponents τ and τ_A.[6] Then, according to Theorem 10.6,

$$\rho = \max(\nu, \tau) = \max(\nu_A, \tau_A).$$

But ρ is not an integer, and hence $\rho \neq \nu$, $\rho \neq \nu_A$, which implies

$$\rho = \tau = \tau_A,$$

as asserted.

Example. Consider the function

$$f(z) = \frac{\sin \sqrt{z}}{\sqrt{z}}$$

of order $\frac{1}{2}$. According to Theorem 10.7, the sequence of A-points of $f(z)$ has convergence exponent $\frac{1}{2}$ for every A. For example, let $A = 0$. Then the A-points of $f(z)$ are given by the sequence

$$\pi^2, (2\pi)^2, \ldots, (n\pi)^2, \ldots,$$

which obviously has convergence exponent $\frac{1}{2}$.

Remark. For a function $f(z)$ of positive integral order ρ, there may exist a value $A = A_0$, called *Borel's exceptional value*, for which the convergence exponent of the sequence of A-points is less than ρ. In particular, any value $A = A_0$ which is taken only a finite number of times, i.e., any Picard exceptional value (see p. 268) is a Borel exceptional value, since then $\tau_A = 0$.[7] For example, the function e^z does not take the value 0 at all, while the function $p(z)e^{P(z)}$, where $p(z)$ and $P(z)$ are polynomials, takes the value 0 a number of times equal to the degree of $p(z)$. However, a Borel exceptional value A_0 may be taken at infinitely many points, and hence need not be a Picard exceptional value. Thus the function $e^{z^2} \sin z$ is of order 2, but its zeros, which coincide with the infinitely many zeros of $\sin z$, have convergence exponent 1, so that 0 is a Borel exceptional value but not a Picard exceptional

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[6] Recall from the proof of Picard's first theorem (Theorem 9.11) that $f(z)$ has infinitely many A-points for every value of A, since ρ is fractional. It follows that $\tau_A > 0$, but Theorem 10.7 tells us much more, i.e., that $\tau_A = \rho$.

[7] It is easy to see that all the results involving infinite sequences and their convergence exponents can be reworded to cover the case of finite sequences as well. We define the convergence exponent of any finite sequence $\zeta_1, \ldots, \zeta_N$ as 0, in keeping with the fact that

$$\sum_{n=1}^{N} \frac{1}{|\zeta_n|^\alpha}$$

is finite for all $\alpha \geqslant 0$.

value. It can be shown (see Problem 10.4) that no entire function of finite order can have two Borel exceptional values. In other words, if $f(z)$ is of positive integral order ρ, then $\tau_A = \rho$ with the possible exception of one value A_0 such that $\tau_{A_0} < \rho$.

50. Meromorphic Functions

It will be recalled that by a *meromorphic function* $f(z)$ we mean a function which can be written as a quotient

$$f(z) = \frac{g(z)}{h(z)}$$

of two entire functions $g(z)$ and $h(z)$, where $(h)z \not\equiv 0$. An equivalent characterization of such functions is given by

THEOREM 10.8. *A single-valued function $f(z)$ is meromorphic if and only if its only singular points in the finite plane are poles.*

Proof. If $f(z)$ has a representation of the form $g(z)/h(z)$ where $g(z)$ and $h(z)$ are entire, then $f(z)$ is analytic at every point where $h(z) \neq 0$. Suppose $h(z)$ has a zero of order $l \geqslant 1$ at a finite point z_0, and suppose z_0 is a zero of order $k \geqslant 0$ of $g(z)$. [If z_0 is not a zero of $g(z)$, we set $k = 0$.] Then, just as on p. 21, $f(z)$ has a pole of order $l - k$ at z_0 if $l > k$ and a regular point at z_0 if $l \leqslant k$, where the regular point is a zero of order $k - l$ if $l < k$. Moreover, since the zeros of $h(z)$ cannot have a finite limit point [otherwise $h(z)$ would vanish identically], neither can the poles of $f(z)$. Thus the only singular points of $f(z)$ in the finite plane are poles, of which there are countably many (by Theorem I.17.3).

Conversely, let $f(z)$ be a function whose only singular points in the finite plane are poles. If $f(z)$ has no poles, then $f(z)$ is entire and hence (trivially) meromorphic. Suppose $f(z)$ has a finite number of poles $b_1, \ldots, b_n$, of orders $\beta_1, \ldots, \beta_n$, respectively. Then, forming the polynomial

$$P(z) = (z - b_1)^{\beta_1} \cdots (z - b_n)^{\beta_n},$$

we see that $g(z) = f(z)P(z)$ is entire. In other words, $f(z)$ can be written as a quotient $g(z)/P(z)$ of two entire functions, as required. Finally, suppose $f(z)$ has an infinite number of poles. Then no compact set F can contain more than a finite number of poles, since otherwise F would contain a limit point of poles, contrary to the assumption that $f(z)$ has no singular points other than poles. Therefore, by the same argument as used to prove Theorem I.17.3, $f(z)$ has countably many poles, which we write as an increasing sequence

$$\underbrace{0, \ldots, 0}_{\lambda \text{ times}}, b_1, \ldots, b_n, \ldots,$$

where $b_n \to \infty$ as $n \to \infty$ (why?), and each pole is repeated a number of times equal to its order.[8] According to Weierstrass' theorem (Theorem 10.1), there is an entire function $h(z)$ whose zeros coincide with the poles of $f(z)$ and have the same orders. But then $g(z) = f(z)h(z)$ is entire, i.e., we can write $f(z)$ as a quotient $g(z)/h(z)$ of two entire functions, and the proof is complete.

As just shown, if a meromorphic function $f(z)$ has infinitely many poles, then the point at infinity is a limit point of these poles, and hence is a non-isolated singular point. On the other hand, if $f(z)$ has only finitely many poles, then ∞ is either a regular point of $f(z)$ or an isolated singular point of single-valued character, i.e., a pole or an essential singular point. The place of rational functions in the class of meromorphic functions is shown by

THEOREM 10.9. *A meromorphic function $f(z)$ is a rational function if and only if the point at infinity is a regular point or a pole of $f(z)$.*

Proof. If $f(z)$ is rational, then ∞ is obviously a regular point or a pole (cf. p. I.161). To prove the converse, we first note that $f(z)$ can have only finitely many poles $b_1, \ldots, b_n$, of orders $\beta_1, \ldots, \beta_n$, since otherwise ∞ would be a limit point of poles, contrary to hypothesis. Let the principal part of $f(z)$ at the pole z_j be

$$G_j(z) = \frac{a_{-\beta_j}}{(z - b_j)^{\beta_j}} + \cdots + \frac{a_{-1}}{z - b_j} \qquad (a_{-\beta_j} \neq 0; j = 1, \ldots, n), \quad (10.32)$$

and let $G_\infty(z)$ denote the polynomial which is the principal part of $f(z)$ at ∞. [If ∞ is a regular point, we set $G_\infty(z) \equiv 0$.] Now consider the function

$$\varphi(z) = f(z) - G_\infty(z) - \sum_{j=1}^{n} G_j(z).$$

Writing $\varphi(z)$ in the form

$$\varphi(z) = f(z) - G_j(z) - [G_\infty(z) + G_1(z) + \cdots + G_{j-1}(z) + G_{j+1}(z) + \cdots + G_n(z)],$$

and noting that the Laurent expansion of $f(z) - G_j(z)$ at z_j contains no negative powers of $z - z_j$, while each of the functions in brackets is analytic at z_j, we find that $\varphi(z)$ is analytic at z_j $(j = 1, \ldots, n)$. Since $\varphi(z)$ is obviously analytic at every finite point $z \neq z_j$, it follows that $\varphi(z)$ is entire. Similarly, writing $\varphi(z)$ in the form

$$\varphi(z) = f(z) - G_\infty(z) - [G_1(z) + \cdots + G_n(z)],$$

[8] This way of writing the poles of $f(z)$ has been chosen to match the statement of Theorem 10.1. If $f(z)$ has no poles at the origin, then $\lambda = 0$.

and noting that the Laurent expansion of $f(z) - G_\infty(z)$ at ∞ contains no positive powers, while each of the functions in brackets is analytic at ∞, we see that $\varphi(z)$ itself is analytic at ∞. But then, according to Liouville's theorem, $\varphi(z)$ must be a constant, i.e.,

$$\varphi(z) = f(z) - G_\infty(z) - \sum_{j=1}^{n} G_j(z) \equiv \text{const} = c,$$

or equivalently,

$$f(z) = c + G_\infty(z) + \sum_{j=0}^{n} G_j(z). \tag{10.33}$$

Therefore $f(z)$ is a rational function, being the sum of a polynomial and a finite number of rational functions, and the theorem is proved.

Remark. Formula (10.33) allows us to represent an arbitrary rational function $f(z)$ as the sum of a polynomial $G_\infty(z)$ and a finite number of rational functions $G_j(z)$ of the special form (10.32). Each of the functions $G_1(z), \ldots, G_n(z), G_\infty(z)$ has just one pole in the extended plane [z_j for $G_j(z)$ and ∞ for $G_\infty(z)$]. The resulting representation of $f(z)$ is called its *partial fraction expansion* (cf. p. 58).

51. Mittag-Leffler's Theorem

Turning to the problem of expanding an *arbitrary* meromorphic function in partial fractions, we now show that any increasing sequence of distinct complex numbers converging to infinity, together with any infinite sequence of rational functions of the form (10.32), can serve as the poles and corresponding principal parts of a meromorphic function $f(z)$:

THEOREM 10.10 (*Mittag-Leffler's theorem*). *Let*

$$\zeta_0 = 0, \zeta_1, \ldots, \zeta_n, \ldots \tag{10.34}$$

be an increasing sequence of distinct complex numbers converging to infinity, and let

$$G_0(z), G_1(z), \ldots, G_n(z), \ldots$$

be a sequence of rational functions of the form

$$G_n(z) = \frac{a_{-\beta_n}^{(n)}}{(z - \zeta_n)^{\beta_n}} + \cdots + \frac{a_{-1}^{(n)}}{z - \zeta_n} \qquad (n = 0, 1, 2, \ldots)$$

where $\beta_n \neq 0$, $a_{-\beta_n}^{(n)} \neq 0$ *if* $n \neq 0$.[9] *Then there exists a meromorphic*

[9] In keeping with the treatment in Secs. 9 and 10, we reserve the symbol ζ_0 for the point $z = 0$, i.e., for a possible pole of $f(z)$ at the origin, with principal part $G_0(z)$. The case where $f(z)$ has no pole at the origin is included by allowing $G_0(z)$ to vanish identically; formally, this corresponds to setting $\beta_0 = 0$, and regarding a regular point as a pole of order zero. Thus $G_n(z)$ has a pole of order β_n at the point ζ_n, but no other poles.

function $f(z)$ whose poles coincide with the points (10.34), *and whose principal part at the pole ζ_n equals $G_n(z)$, for each $n = 0, 1, 2, \ldots$.*

Proof. It is not surprising that the proof bears a strong resemblance to the proof of Weierstrass' theorem (Theorem 10.1). We start from the Taylor series expansion

$$G_n(z) = a_0^{(n)} + a_1^{(n)}z + \cdots + a_k^{(n)}z^k + \cdots \qquad (n = 0, 1, 2, \ldots),$$

which is convergent on the disk $|z| < |\zeta_n|$ and uniformly convergent on every smaller disk, in particular on $D_n: |z| < \frac{1}{2}|\zeta_n|$. Let $\{\varepsilon_n\}$ be any sequence of positive numbers such that

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty. \tag{10.35}$$

Then, choosing integers $k_0, k_1, k_2, \ldots$ such that

$$|G_n(z) - [a_0^{(n)} + a_1^{(n)}z + \cdots + a_{k_n}^{(n)}z^{k_n}]| < \varepsilon_n \qquad (n = 0, 1, 2, \ldots) \tag{10.36}$$

for all $z \in D_n$, we introduce the polynomials

$$P_n(z) = -a_0^{(n)} - a_1^{(n)}z - \cdots - a_{k_n}^{(n)}z^{k_n} \qquad (n = 0, 1, 2, \ldots), \tag{10.37}$$

where $P_0(z) \equiv 0$ if $G_0(z) \equiv 0$. Given any disk $K_R: |z| < R$, let $N(R)$ be the smallest integer such that $|\zeta_n| > 2R$ for all $n > N(R)$. Consider the series

$$\sum_{n=N(R)+1}^{\infty} [G_n(z) + P_n(z)], \tag{10.38}$$

noting that $K_R \subset D_n$ for all $n > N(R)$, while K_R contains none of the points $\zeta_{N(R)+1}, \zeta_{N(R)+2}, \ldots$. It follows from (10.36) and (10.37) that

$$|G_n(z) + P_n(z)| < \varepsilon_n$$

for all $n > N(R)$ and $z \in K_R$. Therefore, because of (10.35) and Weierstrass' M-test (Theorem I.15.2), the series (10.38) is uniformly convergent on K_R, and hence represents an analytic function $\chi_R(z)$ on K_R (cf. Theorem I.15.6).

Thus, if

$$f(z) = \sum_{n=0}^{\infty} [G_n(z) + P_n(z)],$$

we have the representation[10]

$$f(z) = f_{N(R)}(z) + \chi_R(z) \qquad (z \in K_R), \tag{10.39}$$

where $\chi_R(z)$ is analytic on K_R, and the partial sum

$$f_{N(R)}(z) = \sum_{n=0}^{N(R)} [G_n(z) + P_n(z)]$$

[10] Note the analogy between formulas (10.39) and (10.8).

is a rational function whose poles in K_R are precisely those points of the sequence (10.34) which lie in K_R. Moreover, the principal part of $f_{N(R)}(z)$, and hence of $f(z)$, at any point $\zeta_n \in K_R$ is just $G_n(z)$. The theorem now follows at once from the observation that K_R can have arbitrarily large radius.

COROLLARY. *Let $f(z)$ be a meromorphic function whose poles are given by increasing sequence of distinct complex numbers*[11]

$$b_0 = 0, b_1, \ldots, b_n, \ldots,$$

with corresponding principal parts

$$G_0(z), G_1(z), \ldots, G_n(z), \ldots.$$

Then $f(z)$ can be represented in the form

$$f(z) = g(z) + \sum_{n=0}^{\infty} [G_n(z) + P_n(z)],$$

where $g(z)$ is an entire function and the $P_n(z)$ are polynomials.

Proof. Use Mittag-Leffler's theorem to construct a function

$$\varphi(z) = \sum_{n=0}^{\infty} [G_n(z) + P_n(z)]$$

with the same poles and principal parts as $f(z)$. Then $f(z) - \varphi(z)$ is analytic in the whole plane, and hence equals an entire function, which we denote by $g(z)$.

Example 1. *Given an increasing sequence $\{\zeta_n\}$ of distinct nonzero complex numbers converging to infinity, and an arbitrary complex sequence $\{A_n\}$, find an entire function $f(z)$ such that*

$$f(\zeta_n) = A_n \qquad (n = 1, 2, \ldots). \tag{10.40}$$

We begin by using Theorem 10.1, Corollary 1 to construct an entire function $g(z)$ with simple zeros at the points $\zeta_1, \zeta_2, \ldots$, i.e.,

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) \exp\left(\frac{z}{\zeta_n} + \cdots + \frac{z^n}{n\zeta_n^n}\right). \tag{10.41}$$

Then we calculate the derivative $g'(z)$ at every point ζ_n, obtaining a sequence of nonzero complex numbers $\{g'(\zeta_n)\}$. Next we use Mittag-Leffler's theorem to find a meromorphic function $\varphi(z)$, with simple poles at the points $\zeta_1, \zeta_2, \ldots$ and corresponding principal parts

$$\frac{A_n/g'(\zeta_n)}{z - \zeta_n} \qquad (n = 1, 2, \ldots).$$

---

[11] Note that the sequence $\{b_n\}$ necessarily converges to infinity, since $\{b_n\}$ cannot have a finite limit point.

Thus

$$\varphi(z) = \sum_{n=1}^{\infty} \left[\frac{A_n/g'(\zeta_n)}{z - \zeta_n} + P_n(z) \right],$$

where the $P_n(z)$ are suitably chosen polynomials. Then the function

$$f(z) = g(z)\varphi(z)$$

is obviously entire, and moreover satisfies the condition (10.40), since

$$\begin{aligned} f(\zeta_n) &= \lim_{z \to \zeta_n} g(z)\varphi(z) = \lim_{z \to \zeta_n} \left[\frac{g(z) - g(\zeta_n)}{z - \zeta_n} \varphi(z)(z - \zeta_n) \right] \\ &= \frac{g'(\zeta_n)A_n}{g'(\zeta_n)} \qquad (n = 1, 2, \ldots). \end{aligned}$$

Example 2. *Given an increasing sequence $\{\zeta_n\}$ of distinct nonzero complex numbers converging to infinity, find a meromorphic function $f(z)$ with simple poles at the points $\zeta_1, \zeta_2, \ldots$ and corresponding principal parts*

$$\frac{1}{z - \zeta_n} \qquad (n = 1, 2, \ldots). \tag{10.42}$$

We again construct the entire function (10.41) with simple zeros at the points $\zeta_1, \zeta_2, \ldots$, which we now write in the form

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) e^{P_n(z)},$$

where

$$P_n(z) = \frac{z}{\zeta_n} + \cdots + \frac{z^n}{n\zeta_n^n}. \tag{10.43}$$

In the special case where the sequence $\{\zeta_n\}$ has a finite exponent of convergence τ, let $\varkappa$ be the largest nonnegative integer k for which the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^k}$$

diverges. We can then replace the polynomials (10.43) by the polynomials

$$P_n(z) = \frac{z}{\zeta_n} + \cdots + \frac{z^{\varkappa}}{\varkappa\zeta_n^{\varkappa}}, \tag{10.44}$$

which are all of degree $\varkappa$.

In any event, we have

$$\begin{aligned} g(z) &= \prod_{n=1}^{N(R)} \left(1 - \frac{z}{\zeta_n}\right) e^{P_n(z)} \prod_{n=N(R)+1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) e^{P_n(z)} \\ &= g_{N(R)}(z) \prod_{n=N(R)+1}^{\infty} \left(1 - \frac{z}{\zeta_n}\right) e^{P_n(z)}, \end{aligned}$$

where $N(R)$ is the smallest integer such that $|\zeta_n| > 2R$ for all $n > N(R)$. Therefore

$$g(z) = g_{N(R)}(z) \exp\left\{ \sum_{n=N(R)+1}^{\infty} \left[\ln\left(1 - \frac{z}{\zeta_n}\right) + P_n(z) \right] \right\}, \tag{10.45}$$

where the infinite series is uniformly convergent on the disk $K_R: |z| < R$ for the same reason as in the proof of Theorems 10.1 and 10.3. Differentiating (10.45) with respect to z and dividing the result by $g(z)$, we find that

$$\frac{g'(z)}{g(z)} = \frac{g'_{N(R)}(z)}{g_{N(R)}(z)} + \sum_{n=N(R)+1}^{\infty} \left[\frac{1}{z-\zeta_n} + P'_n(z)\right]$$

for all $z \in K_R$ except those points ζ_n which lie in K_R.[12] But obviously

$$\frac{g'_{N(R)}(z)}{g_{N(R)}(z)} = \sum_{n=1}^{N(R)} \left[\frac{1}{z-\zeta_n} + P'_n(z)\right],$$

and hence

$$f(z) = \frac{g'(z)}{g(z)} = \sum_{n=1}^{\infty} \left[\frac{1}{z-\zeta_n} + P'_n(z)\right]. \tag{10.46}$$

Since the disk K_R can have arbitrarily large radius, $f(z)$ is a meromorphic function with poles $\zeta_1, \zeta_2, \ldots$ and principal parts (10.42), as required. Here the polynomials $P'_n(z)$ guaranteeing the convergence of the series (10.46) have the particularly simple form

$$P'_n(z) = \frac{1}{\zeta_n} + \frac{z}{\zeta_n^2} + \cdots + \frac{z^{n-1}}{\zeta_n^n}$$

or

$$P'_n(z) = \frac{1}{\zeta_n} + \frac{z}{\zeta_n^2} + \cdots + \frac{z^{\varkappa-1}}{\zeta_n^{\varkappa}},$$

depending on whether $P_n(z)$ is of the form (10.43) or (10.44). Thus the most general function solving the given problem can be written in the form

$$f(z) = \varphi(z) + \sum_{n=1}^{\infty} \left[\frac{1}{z-\zeta_n} + \frac{1}{\zeta_n} + \frac{z}{\zeta_n^2} + \cdots + \frac{z^{\nu_n-1}}{\zeta_n^{\nu_n}}\right], \tag{10.47}$$

where $\varphi(z)$ is an entire function and $\nu_n = \varkappa$ or $\nu_n = n$, depending on whether or not the sequence $\{\zeta_n\}$ has a finite exponent of convergence.

Example 3. *Given an increasing sequence $\{\zeta_n\}$ of distinct nonzero complex numbers converging to infinity, find a meromorphic function $f(z)$ with poles of order 2 at the points $\zeta_1, \zeta_2, \ldots$ and corresponding principal parts*

$$\frac{1}{(z-\zeta_n)^2} \qquad (n = 1, 2, \ldots).$$

Replacing f by F and φ by Φ, we differentiate (10.47) term by term with respect to z.[13] The result is

$$F'(z) = \Phi'(z) + \sum_{n=1}^{\infty} \left[-\frac{1}{(z-\zeta_n)^2} + \frac{1}{\zeta_n^2} + \frac{2z}{\zeta_n^3} + \cdots + \frac{(\nu_n-1)z^{\nu_n-2}}{\zeta_n^{\nu_n}}\right],$$

---

[12] The term-by-term differentiation of the infinite series is justified by its uniform convergence on K_R (cf. Theorem I.15.6).

[13] The term-by-term differentiation of the infinite series (10.47) is justified by its uniform convergence on every disk K_R, if we omit a finite number of terms with poles in K_R.

or

$$f(z) = \varphi(z) + \sum_{n=1}^{\infty} \left[\frac{1}{(z - \zeta_n)^2} - \frac{1}{\zeta_n^2} - \frac{2z}{\zeta_n^3} - \cdots - \frac{(\nu_n - 1)z^{\nu_n - 2}}{\zeta_n^{\nu_n}} \right], \quad (10.48)$$

after replacing $-F'$ by f and Φ' by φ. In particular, suppose the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^3}$$

converges, which allows us to set $\nu_n = 2$, whether or not the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^2}$$

diverges. Then (10.48) simplifies to

$$f(z) = \varphi(z) + \sum_{n=1}^{\infty} \left[\frac{1}{(z - \zeta_n)^2} - \frac{1}{\zeta_n^2} \right].$$

Remark 1. Of course, the results of Examples 2 and 3 can be obtained directly from Mittag-Leffler's theorem, without recourse to Weierstrass' theorem (see Problem 10.9).

Remark 2. Other examples of partial fraction expansions, derived by a different method, have already been encountered in Sec. 10.

52. The Gamma Function

The *gamma function* $\Gamma(z)$, first introduced by Euler, is the simplest and most important of an infinite set of meromorphic functions which extend the concept of the factorial function $n!$ (originally defined only for positive integral n) to the case of arbitrary complex z. For historical reasons, the relation between $\Gamma(z)$ and $n!$ is given by the formula

$$\Gamma(n + 1) = n!,$$

rather than the more natural formula $\Gamma(n) = n!$, which would allow $\Gamma(z)$ to be thought of as $z!$ Thus the relation

$$n(n - 1)! = n!,$$

characterizing the factorial function, takes the form

$$n\Gamma(n) = \Gamma(n + 1) \quad (10.49)$$

in terms of the gamma function. In particular, substituting $n = 1$ into (10.49), we find that[14]

$$\Gamma(1) = \Gamma(2) = 1! = 1.$$

[14] Since $\Gamma(1)$ is formally equal to $0!$, this justifies the familiar formula $0! = 1$.

These considerations suggest that we study the solutions of the functional equation

$$zf(z) = f(z + 1) \qquad [f(1) = 1] \tag{10.50}$$

for complex z. The solution of (10.50) must be a meromorphic function, as can be seen by successively applying (10.50) to the values $z, z + 1, \ldots, z + n - 1$ (where n is a positive integer), and then combining the results to obtain the formula

$$z(z + 1)\cdots(z + n - 1)f(z) = f(z + n). \tag{10.51}$$

In fact, letting z approach $-(n - 1) = -m$ $(m = 0, 1, \ldots)$ in (10.51), we find that

$$\lim_{z \to -m} (z + m)f(z) = \frac{f(1)}{(-1)^m m!} = \frac{(-1)^m}{m!},$$

i.e., $f(z)$ has simple poles at the points

$$z = -m \qquad (m = 0, 1, \ldots), \tag{10.52}$$

with corresponding residues

$$\frac{(-1)^m}{m!}.$$

By itself, equation (10.50) cannot be used to define the gamma function, since the solution of (10.50) is not unique. To see this, let $\varphi(z) \not\equiv 1$ be any meromorphic function such that[15]

$$\varphi(z + 1) = \varphi(z) \qquad [\varphi(1) = 1]. \tag{10.53}$$

Then the function

$$f^*(z) = \varphi(z)f(z) \tag{10.54}$$

also satisfies (10.50). By imposing the extra requirement that $f(z)$ have no poles other than the points (10.52) and no zeros at all, we can narrow the class of functions satisfying (10.50). For example, if $f(z)$ satisfies (10.50), so does the function

$$f^*(z) = \frac{\tan(i + 2\pi z)}{\tan i} f(z),$$

but $f^*(z)$ has infinitely many (imaginary) zeros introduced by the factor $\tan(i + 2\pi z)$. However, even this additional requirement is not enough to make the solution of (10.50) unique. In fact, this time let $\varphi(z) \not\equiv 1$ be any nonvanishing entire function satisfying (10.53).[16] Then, if $f(z)$ is a solution of (10.50), so is the function (10.54). The following theorem shows the extent

[15] Thus $\varphi(z)$ is periodic, with period 1.

[16] Such a function is $\varphi(z) = e^{2\pi i z}$.

to which the solution of (10.50) remains arbitrary:

THEOREM 10.11. *Let $f(z)$ be a meromorphic function satisfying* (10.50), *with no zeros and no poles other than the points $z = -m$ $(m = 0, 1, \ldots)$. Then $f(z)$ can be represented in the form*

$$f(z) = e^{-g(z)} \frac{1}{z \prod_{m=1}^{\infty} \left(1 + \frac{z}{m}\right) e^{-z/m}}, \tag{10.55}$$

in terms of an entire function $g(z)$ such that

$$g(z + 1) - g(z) = C + 2k\pi i \qquad [g(1) = C + 2l\pi i], \tag{10.56}$$

where k and l are integers, and C is Euler's constant, defined by[17]

$$C = \lim_{n \to \infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \ln n \right) = 0.5772\ldots \tag{10.57}$$

Proof. Since $f(z)$ has no zeros, and no poles except at the points $z = -m$ $(m = 0, 1, \ldots)$, the function

$$F(z) = \frac{1}{f(z)}$$

is entire, with its only zeros at the same points. Since 1 is the largest nonnegative integer k for which

$$\sum_{m=1}^{\infty} \frac{1}{m^k}$$

diverges, it follows from Theorem 10.3 that $F(z)$ has the representation

$$F(z) = e^{g(z)} z \prod_{m=1}^{\infty} \left(1 + \frac{z}{m}\right) e^{-z/m},$$

where $g(z)$ is an entire function. Recalling the meaning of $F(z)$, we obtain (10.55). To see what (10.50) implies about $g(z)$, we write (10.55) in the form

$$f(z) = \lim_{n \to \infty} f_n(z), \tag{10.58}$$

where

$$f_n(z) = \frac{e^{-g(z)}}{z \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right) e^{-z/m}} = \frac{n! \exp\left[-g(z) + \sum_{m=1}^{n} \frac{z}{m}\right]}{z(z + 1) \cdots (z + n)}. \tag{10.59}$$

[17] For a proof that the limit (10.57) exists, see e.g., D. V. Widder, *op. cit.*, p. 387.

Then

$$\begin{aligned}\frac{zf(z)}{f(z+1)} &= \lim_{n\to\infty}\frac{zf_n(z)}{f_n(z+1)}\\ &= \lim_{n\to\infty}(z+n+1)\exp\left[-g(z)+g(z+1)-\sum_{m=1}^{n}\frac{1}{m}\right]\\ &= \lim_{n\to\infty}\left(1+\frac{z+1}{n}\right)\exp\left[-g(z)+g(z+1)-\left(\sum_{m=1}^{n}\frac{1}{m}-\ln n\right)\right]\\ &= \exp\left[-g(z)+g(z+1)-C\right]\end{aligned}$$

for every z, where C is given by (10.57), and moreover

$$\begin{aligned}f(1) = \lim_{n\to\infty} f_n(1) &= \lim_{n\to\infty}\frac{\exp\left[-g(1)+\sum_{m=1}^{n}\left(\frac{1}{m}-\ln n\right)\right]}{1+\frac{1}{n}}\\ &= \exp\left[-g(1)+C\right].\end{aligned}$$

Therefore (10.50) implies

$$\exp\left[-g(z)+g(z+1)-C\right]=1, \qquad \exp\left[-g(1)+C\right]=1,$$

or equivalently (10.56), and the theorem is proved.

The simplest entire function $g(z)$ satisfying (10.56) is obviously the linear function $g(z)=Cz$, and we complete our definition of the gamma function by making precisely this choice. In other words, by the gamma function $\Gamma(z)$ we mean the meromorphic function

$$\Gamma(z) = e^{-Cz}\frac{1}{z\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)e^{-z/m}}, \tag{10.60}$$

whose reciprocal is the entire function

$$\frac{1}{\Gamma(z)} = e^{Cz}z\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)e^{-z/m}. \tag{10.61}$$

Since the zeros of (10.61) have convergence exponent 1, and since the polynomial Cz is of degree 1, it follows from Borel's theorem (Theorem 10.6) that (10.61) is an entire function of order 1. In Figure 10.1 we plot the modular surface of the function $\Gamma(z)$, i.e., the surface $\mu = |\Gamma(z)|$. The curves lying in horizontal planes belong to the family $\mu = \text{const}$, while the curves labelled in degrees belong to the family $\arg\Gamma(z) = \text{const}$. Similarly, Figure 10.2 shows the modular surface of the function $1/\Gamma(z)$.

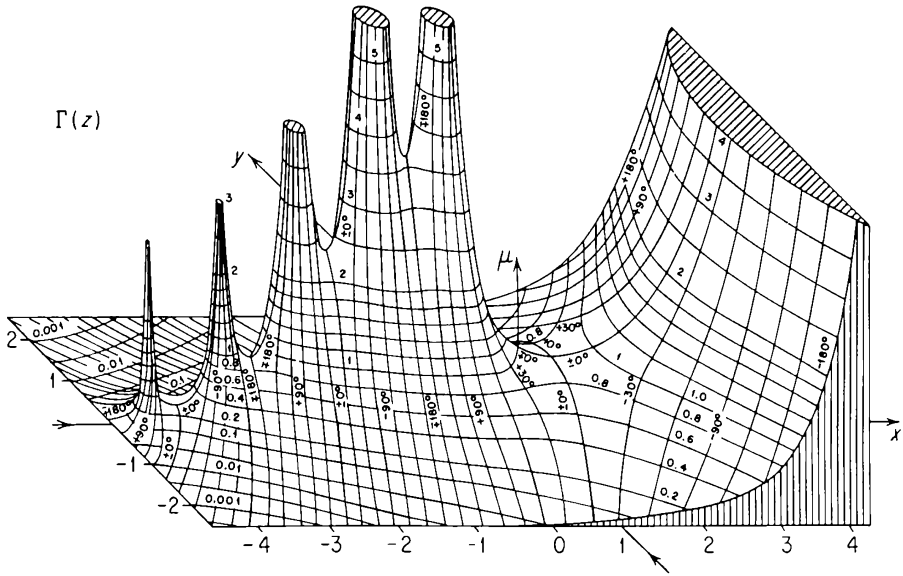

FIGURE 10.1

It follows from (10.61) and (10.24) that

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right) = -\frac{z}{\pi}\pi z \prod_{m=1}^{\infty} \left(1 - \frac{\pi^2 z^2}{m^2 \pi^2}\right) = -\frac{z \sin \pi z}{\pi}$$

or

$$\frac{1}{\Gamma(z)[-z\Gamma(-z)]} = \frac{\sin \pi z}{\pi}. \tag{10.62}$$

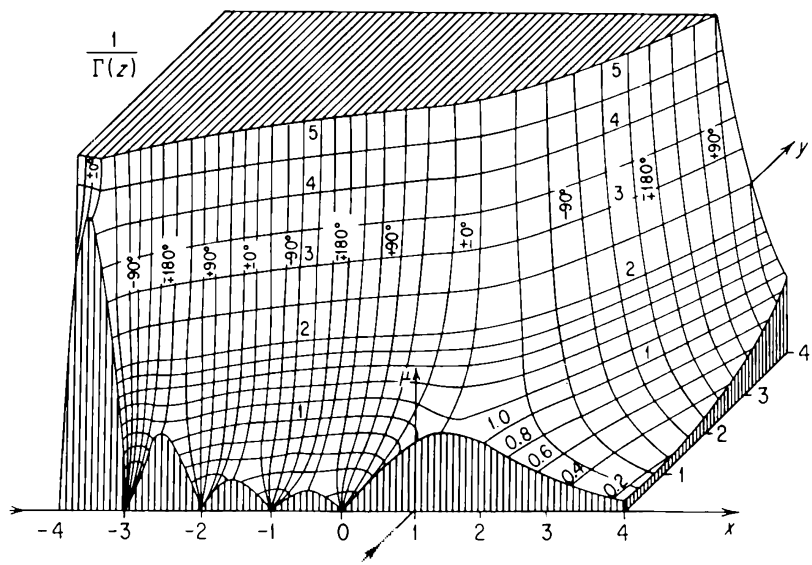

FIGURE 10.2

But according to (10.50), with $f(z)$ replaced by $\Gamma(z)$,

$$z\Gamma(z) = \Gamma(z + 1), \tag{10.63}$$

and hence

$$-z\Gamma(-z) = \Gamma(1 - z).$$

Therefore (10.62) becomes

$$\frac{1}{\Gamma(z)\Gamma(1 - z)} = \frac{\sin \pi z}{\pi}$$

or

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}, \tag{10.64}$$

an important formula in the theory of the gamma function. In particular, (10.64) implies

$$[\Gamma(\tfrac{1}{2})]^2 = \pi$$

or

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \tag{10.65}$$

since $\Gamma(\frac{1}{2}) > 0$, according to (10.60). Together (10.63) and (10.65) imply

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdots (2n - 1)}{2^n} \sqrt{\pi} \qquad (n = 1, 2, \ldots).$$

Finally, replacing $g(z)$ by Cz and $f(z)$ by $\Gamma(z)$ in (10.58) and (10.59), we find that

$$\begin{aligned} \Gamma(z) &= \lim_{n \to \infty} \frac{n! \exp\left\{\left(\sum_{m=1}^{n} \frac{1}{m} - C\right)z\right\}}{z(z + 1) \cdots (z + n)} \\ &= \lim_{n \to \infty} \frac{n! \exp\left\{\left[\left(\sum_{m=1}^{n} \frac{1}{m} - \ln n - C\right) + \ln n\right]z\right\}}{z(z + 1) \cdots (z + n)}. \end{aligned} \tag{10.66}$$

But according to (10.57),

$$\lim_{n \to \infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \ln n - C\right) = 0,$$

while obviously

$$\exp(z \ln n) = n^z.$$

Therefore (10.66) reduces to the following basic representation of the gamma function:

$$\Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z(z + 1) \cdots (z + n)}. \tag{10.67}$$

*53. Integral Representations of $\Gamma(z)$. Partial Fraction Expansion of $\Gamma(z)$

Consider the function

$$f(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad (\mathrm{Re}\, z > 0), \tag{10.68}$$

where the path of integration is along the positive real axis, and t^{z-1} denotes the function $\exp[(z-1)\ln t]$. Suppose we write (10.68) in the form

$$f(z) = \int_0^1 e^{-t} t^{z-1}\, dt + \int_1^\infty e^{-t} t^{z-1}\, dt. \tag{10.69}$$

Since

$$|e^{-t} t^{z-1}| = e^{-t} t^{\mathrm{Re}\, z - 1},$$

the first integral in (10.69) is absolutely convergent if z belongs to the right half-plane $D: \mathrm{Re}\, z > 0$, while the second integral is absolutely convergent for all z. It follows from Theorems I.17.20 and I.17.21 that the first integral represents a function analytic on D, while the second integral represents an entire function. Therefore $f(z)$ itself is analytic on D, which explains the condition $\mathrm{Re}\, z > 0$ in (10.68).

THEOREM 10.12. *The function* (10.68) *coincides with* $\Gamma(z)$ *in the half-plane* $D: \mathrm{Re}\, z > 0$, *i.e.*,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad (\mathrm{Re}\, z > 0). \tag{10.70}$$

Proof. First we prove that

$$\lim_{n \to \infty} f_n(z) = f(z) \qquad (z \in D), \tag{10.71}$$

where

$$f_n(z) = \int_0^n \left(1 - \frac{t}{z}\right)^n t^{z-1}\, dt \qquad (n = 1, 2, \ldots) \tag{10.72}$$

and $f(z)$ is given by (10.68). [Note that according to Theorem I.17.21, every $f_n(z)$ is analytic on D.] This is the same as proving that

$$\lim_{n \to \infty} \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt = 0, \tag{10.73}$$

since

$$f(z) = \lim_{n \to \infty} \int_0^n e^{-t} t^{z-1}\, dt,$$

by definition. If $|t| < n$, then

$$1 + \frac{t}{n} \leqslant e^{t/n} \leqslant \frac{1}{1 - \dfrac{t}{n}},$$

and hence

$$\left(1 + \frac{t}{n}\right)^n \leqslant e^t, \qquad \left(1 - \frac{t}{n}\right)^n \leqslant e^{-t},$$

which implies

$$0 \leqslant e^{-t} - \left(1 - \frac{t}{n}\right)^n = e^{-t}\left[1 - e^t\left(1 - \frac{t}{n}\right)^n\right] \leqslant e^{-t}\left[1 - \left(1 - \frac{t^2}{n^2}\right)^n\right]$$
$$= e^{-t}\frac{t^2}{n^2}\left[1 + \left(1 - \frac{t^2}{n^2}\right) + \cdots + \left(1 - \frac{t^2}{n^2}\right)^{n-1}\right] \leqslant \frac{t^2 e^{-t}}{n}.$$

Therefore

$$\left|\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\, dt\right| < \frac{1}{n}\int_0^n e^{-t}t^{x+1}\, dt < \frac{1}{n}\int_0^\infty e^{-t}t^{x+1}\, dt,$$

where $x = \operatorname{Re} z$, and (10.71) follows at once.

Next we show that

$$\lim_{n\to\infty} f_n(z) = \Gamma(z) \qquad (z \in D). \tag{10.74}$$

The substitution $t = n\tau$ transforms (10.72) into

$$f_n(z) = n^z \int_0^1 (1 - \tau)^n \tau^{z-1}\, d\tau. \tag{10.75}$$

Integrating (10.75) by parts n times, we obtain

$$f_n(z) = \frac{n!\, n^z}{z(z+1)\cdots(z+n-1)}\int_0^1 \tau^{z+n-1}\, d\tau = \frac{n!\, n^z}{z(z+1)\cdots(z+n)},$$

which, together with (10.67), implies (10.74). The representation (10.70) now follows at once by comparing (10.71) and (10.74), and recalling the definition of $f(z)$.

Theorem 10.12 has the following important implication:

THEOREM 10.13. *The partial fraction expansion of* $\Gamma(z)$ *is given by the formula*

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z+n)} + \int_1^\infty e^{-t}t^{z-1}\, dt. \tag{10.76}$$

Proof. According to (10.70),

$$\Gamma(z) = \int_0^1 e^{-t}t^{z-1}\, dt + \int_1^\infty e^{-t}t^{z-1}\, dt \qquad (\operatorname{Re} z > 0). \tag{10.77}$$

Let $\varphi(z)$ denote the first integral in the right-hand side of (10.77). Then replacing e^{-t} by its power series expansion and integrating term by term (why is this permissible?), we find that

$$\varphi(z) = \int_0^1 e^{-t}t^{z-1}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\int_0^1 t^{n+z-1}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z+n)} \tag{10.78}$$

if $z \neq 0, -1, -2, \ldots$. Substitution of (10.78) into (10.77) gives (10.76), with the extra condition $\operatorname{Re} z > 0$, which is easily eliminated. In fact, $\varphi(z)$ is a meromorphic function with the same poles and residues as $\Gamma(z)$, and hence the same principal parts as $\Gamma(z)$, since all the poles are simple.[18] It follows that $\Gamma(z) - \varphi(z)$ is entire. But $\Gamma(z) - \varphi(z)$ equals the integral

$$\int_1^\infty e^{-t} t^{z-1}\, dt \tag{10.79}$$

in the half-plane $\operatorname{Re} z > 0$, and as already noted, (10.79) represents an entire function. Therefore $\Gamma(z) - \varphi(z)$ equals (10.79) *for all* z, and the proof is complete.

An interesting integral representation of the entire function $1/\Gamma(z)$ is given by

THEOREM 10.14. *Let* L_ε *denote the contour obtained by traversing in turn the negative real axis from* $-\infty$ *to* $-\varepsilon$, *the circle* $|z| = \varepsilon$ *once in the counterclockwise direction, starting and ending at the point* $-\varepsilon$, *and finally the negative real axis from* $-\varepsilon$ *to* $-\infty$. *Then*

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{L_\varepsilon} e^t t^{-z}\, dt \tag{10.80}$$

for all z *and* $\varepsilon > 0$, *where* $t^{-z} = \exp(-z \ln t)$.

Proof. Let $L_{\varepsilon,n}$ be the part of the contour L_ε contained in the closed disk $|z| \leqslant n$. (The curve L_ε is shown schematically in Figure 10.3, and the heavier part corresponds to $L_{\varepsilon,n}$.) On the lower edge of $L_{\varepsilon,n}$ (the part going from $-n$ to $-\varepsilon$), we have

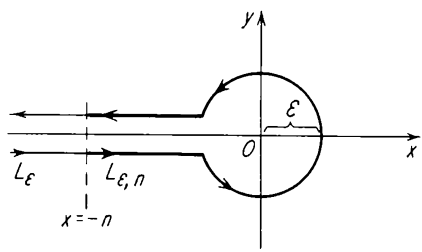

FIGURE 10.3

$$t^{-z} = \exp(-z \ln |t| + i\pi z),$$

while on the upper part of $L_{\varepsilon,n}$ (the part going from $-\varepsilon$ to $-n$), we have[19]

$$t^{-z} = \exp(-z \ln |t| - i\pi z).$$

Under these conditions, each of the integrals

$$\psi_n(z) = \int_{L_{\varepsilon,n}} e^t t^{-z}\, dt \qquad (n = 1, 2, \ldots)$$

[18] Note that the right-hand side of (10.78) is a uniformly convergent series of analytic functions on every compact set containing none of the points $0, -1, -2, \ldots$.

[19] These considerations can be made somewhat more rigorous by passing to the limit $\alpha \to 0$ in the integral studied in Problem 10.15.

represents an entire function (apply Theorem I.17.19). Moreover, the sequence $\{\psi_n(z)\}$ is uniformly convergent on every compact set, since if $|z| < R$,

$$\begin{aligned}|\psi_{n+p}(z) - \psi_n(z)| &\leqslant \left|\int_{-(n+p)}^{-n} \exp(t - z\ln|t| + i\pi z)\,dt\right| \\ &\quad + \left|\int_{-(n+p)}^{-n} \exp(t - z\ln|t| - i\pi z)\,dt\right| \\ &\leqslant 2\int_{-(n+p)}^{-n} \exp(t - x\ln|t| + \pi|y|)\,dt \\ &= 2\int_{n}^{n+p} e^{-t}t^{-x}e^{\pi|y|}\,dt \\ &< 2e^{\pi R}\int_{n}^{n+p} e^{-t}t^{R}\,dt.\end{aligned}$$

Therefore

$$\psi(z) = \lim_{n\to\infty} \psi_n(z)$$

is an entire function, represented by the improper integral

$$\psi(z) = \int_{L_\varepsilon} e^t t^{-z}\,dt. \tag{10.81}$$

To verify that (10.81) is independent of ε, we note that if $0 < \eta < \varepsilon$, then

$$\int_{L_\varepsilon} e^t t^{-z}\,dt - \int_{L_\eta} e^t t^{-z}\,dt = \int_{\Lambda} e^t t^{-z}\,dt,$$

where Λ is the closed contour consisting of the two circles $|z| = \varepsilon$ and $|z| = \eta$, and the segment $-\varepsilon \leqslant t \leqslant -\eta$ traversed twice in opposite directions. But, according to Cauchy's integral theorem, the integral

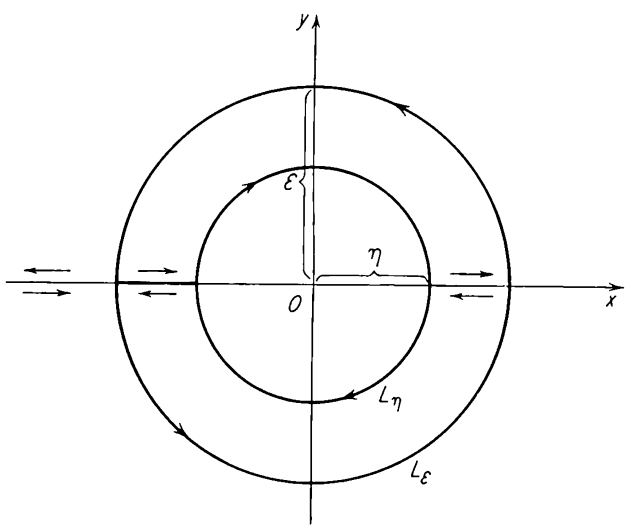

FIGURE 10.4

along Λ vanishes. One way of seeing this is to think of Λ as the sum of two contours shaped like "half-rings" (the parts into which Λ is divided by the positive real axis), and then apply Cauchy's integral theorem to each half-ring separately (see Figure 10.4).

Now suppose that

$$\operatorname{Re} z = x < 1, \quad \varepsilon < 1, \quad t = \varepsilon e^{i\theta}.$$

Then the inequality

$$|e^t t^{-z}| = \exp\left(\varepsilon \cos 0 + x \ln \frac{1}{\varepsilon} + 0y\right) < \frac{1}{\varepsilon^x} e^{1+\pi|y|}$$

implies that

$$\left|\int_{|t|=\varepsilon} e^t t^{-z}\, dt\right| < 2\pi\varepsilon^{1-x} e^{1+\pi|y|} \to 0$$

as $\varepsilon \to 0$. Therefore, passing to the limit $\varepsilon \to 0$ in (10.81), we can write $\psi(z)$ in the form

$$\psi(z) = \int_{L_0} e^t t^{-z}\, dt \qquad (\operatorname{Re} z < 1),$$

where L_0 denotes the negative real axis traversed twice in opposite directions. More exactly, we have

$$\begin{aligned}\psi(z) &= \int_{-\infty}^{0} \exp(t - z \ln |t| + i\pi z)\, dt - \int_{-\infty}^{0} \exp(t - z \ln |t| - i\pi z)\, dt \\ &= e^{i\pi z} \int_0^\infty e^{-t} t^{-z}\, dt - e^{-i\pi z} \int_0^\infty e^{-t} t^{-z}\, dt \\ &= (e^{i\pi z} - e^{-i\pi z}) \int_0^\infty e^{-t} t^{(1-z)-1}\, dt.\end{aligned} \tag{10.82}$$

Since $\operatorname{Re} z < 1$, we have $\operatorname{Re}(1 - z) > 0$, and therefore, according to (10.70), the last integral on the right equals $\Gamma(1 - z)$, i.e., (10.82) becomes

$$\psi(z) = 2i \sin \pi z \Gamma(1 - z). \tag{10.83}$$

Using (10.64), we transform (10.83) into

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \psi(z). \tag{10.84}$$

So far this formula holds only for $\operatorname{Re} z < 1$. However, both sides of (10.84) are entire, and hence the formula holds *for all* z. Replacing $\psi(z)$ by the integral representation (10.81), which is also valid for all z, we finally obtain formula (10.80), and the proof is complete.

Remark. Replacing z by $1 - z$ in (10.80), and using (10.64), we obtain the integral representation

$$\Gamma(z) = \frac{1}{2i \sin \pi z} \int_{L_\varepsilon} e^t t^{z-1}\, dt. \tag{10.85}$$

*54. Asymptotic Behavior of $\Gamma(z)$. Stirling's Formula

We begin by proving an elementary relation between sums and integrals:

LEMMA 1. *Let $f(t)$ be a continuously differentiable function, defined for all $t \geqslant 0$. Then*

$$\sum_{m=0}^{n} f(m) - \int_0^n f(t)\,dt = \frac{f(0) + f(n)}{2} + \int_0^n f'(t)(\{t\} - \tfrac{1}{2})\,dt, \quad (10.86)$$

where $\{t\} = t - [t]$ denotes the fractional part of t.[20]

Proof. Obviously

$$\begin{aligned} f(m) - \int_{m-1}^{m} f(t)\,dt &= f(m) - \int_{m-1}^{m} f(t)\,d(t - m + \tfrac{1}{2}) \\ &= f(m) - \frac{f(m) + f(m-1)}{2} + \int_{m-1}^{m} f'(t)(t - m + \tfrac{1}{2})\,dt \end{aligned} \quad (10.87)$$

for any positive integer m. If $m - 1 \leqslant t < m$, then $m = [t] + 1$, and hence

$$t - m + \tfrac{1}{2} = \{t\} - \tfrac{1}{2}, \quad (10.88)$$

since $t = [t] + \{t\}$. It is clear that $\{t\} - \frac{1}{2}$ is a periodic function, with period 1, which is continuous for all nonintegral t and satisfies the inequality

$$-\tfrac{1}{2} \leqslant \{t\} - \tfrac{1}{2} < \tfrac{1}{2}.$$

Substituting (10.88) into the last integral in the right-hand side of (10.87),[21] we find that

$$\begin{aligned} f(m) - \int_{m-1}^{m} f(t)\,dt \\ = \frac{f(m) - f(m-1)}{2} + \int_{m-1}^{m} f'(t)(\{t\} - \tfrac{1}{2})\,dt. \end{aligned} \quad (10.89)$$

Successively setting $m = 1, 2, \ldots, n$ in (10.89), and adding the resulting equations, we obtain

$$\sum_{m=1}^{n} f(m) - \int_0^n f(t)\,dt = \frac{f(n) - f(0)}{2} + \int_0^n f'(t)(\{t\} - \tfrac{1}{2})\,dt,$$

which immediately implies (10.86).

Lemma 1 leads to the following important representation of the logarithm

---

[20] Recall that $[t]$ is the integral part of t.

[21] The fact that the two expressions $t - m + \frac{1}{2}$ and $\{t\} - \frac{1}{2}$ differ at the single point $t = m$ has no effect on the value of the integral.

of the gamma function:

LEMMA 2. *Let* $\varphi(t)$ *be the function*

$$\varphi(t) = \tfrac{1}{2}\{t\}(\{t\} - 1). \tag{10.90}$$

Then

$$\ln \frac{1}{\Gamma(z)} = (z - 1) - (z - \tfrac{1}{2}) \ln z + \int_0^\infty \frac{\varphi(t)}{(z + t)^2}\, dt - \int_0^\infty \frac{\varphi(t)}{(1 + t)^2}\, dt + 2\pi i k(z), \tag{10.91}$$

where the function $k(z)$ *takes only integral values.*

Proof. Starting from (10.67) and noting that

$$\lim_{n \to \infty} [\ln (n + 1) - \ln n] = \lim_{n \to \infty} \ln \left(1 + \frac{1}{n}\right) = 0,$$

we find that

$$\ln \frac{1}{\Gamma(z)} = \lim_{n \to \infty} \left\{ \sum_{m=0}^{n} [\ln (z + m) - \ln (1 + m)] - (z - 1) \ln n + 2\pi i k_n(z) \right\}, \tag{10.92}$$

where $k_n(z)$ is the integer needed to assure that the argument θ of the right-hand side takes a value in the interval $-\pi < \theta \leqslant \pi$. Assuming that z is fixed and does not belong to the negative real axis (including the origin), we apply formula (10.86) to the function $\ln (z + t)$, obtaining

$$\sum_{m=0}^{n} \ln (z + m) - \int_0^n \ln (z + t)\, dt = \frac{\ln z + \ln (z + n)}{2} + \int_0^n \frac{\{t\} - \frac{1}{2}}{z + t}\, dt$$

or

$$\sum_{m=0}^{n} \ln (z + m) = (z + n + \tfrac{1}{2}) \ln (z + n) - (z - \tfrac{1}{2}) \ln z - n + \int_0^n \frac{\{t\} - \frac{1}{2}}{z + t}\, dt, \tag{10.93}$$

where we have used the formula

$$\int_0^n \ln (z + t)\, dt = (z + n) \ln (z + n) - z \ln z - n.$$

Setting $z = 1$ in (10.93) and subtracting the resulting equation from (10.93) itself, we arrive at the formula

$$\sum_{m=0}^{n} [\ln (z + m) - \ln (1 + m)] = (z + n + \tfrac{1}{2}) \ln (z + n) - (n + \tfrac{3}{2}) \ln (1 + n) - (z - \tfrac{1}{2}) \ln z + \int_0^n \frac{\{t\} - \frac{1}{2}}{z + t}\, dt - \int_0^n \frac{\{t\} - \frac{1}{2}}{1 + t}\, dt,$$

which, together with (10.92), implies

$$\ln \frac{1}{\Gamma(z)} = \lim_{n\to\infty} [(z + n + \tfrac{1}{2}) \ln (z + n) - (n + \tfrac{3}{2}) \ln (1 + n) - (z - \tfrac{1}{2}) \ln z - (z - 1) \ln n + J_n(z) - J_n(1) + 2\pi i k_n(z)],$$

where (10.94)

$$J_n(z) = \int_0^n \frac{\{t\} - \frac{1}{2}}{z + t}\, dt. \tag{10.95}$$

The terms in the right-hand side of (10.94) involving logarithms can be combined to give

$$\begin{aligned}(z-1)[\ln (z+n) - \ln n] + (n+\tfrac{3}{2})[\ln (z+n) - \ln (1+n)] - (z-\tfrac{1}{2}) \ln z \\ = (z-1) \int_n^{z+n} \frac{dz}{z} + (n+\tfrac{3}{2}) \int_{1+n}^{z+n} \frac{dz}{z} - (z-\tfrac{1}{2}) \ln z \\ = (z-1) \int_1^{1+z} \frac{d\zeta}{\zeta + n - 1} + \int_1^z \frac{n+\frac{3}{2}}{\zeta + n}\, d\zeta - (z-\tfrac{1}{2}) \ln z,\end{aligned}$$

where the paths of integration can be chosen to be line segments, say. It follows that the logarithmic terms in (10.94) approach

$$(z - 1) - (z - \tfrac{1}{2}) \ln z$$

as $n \to \infty$, i.e.,

$$\ln \frac{1}{\Gamma(z)} = (z - 1) - (z - \tfrac{1}{2}) \ln z + \lim_{n\to\infty} J_n(z) - \lim_{n\to\infty} J_n(1) + 2\pi i k(z), \tag{10.96}$$

where

$$k(z) = \lim_{n\to\infty} k_n(z)$$

takes only integral values.

Next we introduce the function

$$\varphi(t) = \int_0^t (\{t\} - \tfrac{1}{2})\, dt,$$

which, as anticipated by the notation, is identical with (10.90). In fact, since

$$\int_{m-1}^m (\{t\} - \tfrac{1}{2})\, dt = \int_{m-1}^m (t - m + \tfrac{1}{2})\, dt = 0$$

[recall (10.88)], we have

$$\begin{aligned}\varphi(t) = \int_0^t (\{t\} - \tfrac{1}{2})\, dt &= \int_{[t]}^t (\{t\} - \tfrac{1}{2})\, dt \\ &= \int_0^{\{t\}} (\tau - \tfrac{1}{2})\, d\tau = \tfrac{1}{2}\{t\}(\{t\} - 1).\end{aligned}$$

Clearly, $\varphi(t)$ is a continuous periodic function with period 1, taking real values in the interval

$$-\tfrac{1}{8} \leqslant \varphi(t) \leqslant 0. \tag{10.97}$$

Integrating the right-hand side of (10.95) by parts, we find that

$$J_n(z) = \frac{\varphi(t)}{z+t}\Big|_{t=0}^{t=n} + \int_0^n \frac{\varphi(t)}{(z+t)^2}\,dt = \int_0^n \frac{\varphi(t)}{(z+t)^2}\,dt,$$

and hence

$$\lim_{n\to\infty} J_n(z) = \int_0^\infty \frac{\varphi(t)}{(z+t)^2}\,dt,$$

which, when compared with (10.96), gives (10.91), thereby proving the lemma.

We are now in a position to prove

THEOREM 10.15. *Given any number* δ *in the interval* (0, 1), *let* G_δ *be the half-strip of width* 2δ *defined by the inequalities* $x < \delta$, $|y| < \delta$.[22]

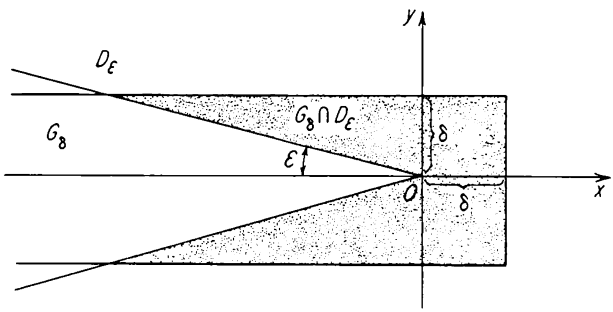

FIGURE 10.5

Then, if $z \notin G_\delta$,

$$\ln \frac{1}{|\Gamma(z)|} = x - 1 - (x - \tfrac{1}{2}) \ln |z| + y \arg z + C(z), \quad (10.98)$$

where $|C(z)| < \pi/4\delta$.

Proof. According to (10.97), $|\varphi(t)| \leqslant \frac{1}{8}$ for all t, and hence

$$\left|\int_0^\infty \frac{\varphi(t)}{(z+t)^2}\,dt\right| \leqslant \frac{1}{8}\int_0^\infty \frac{dt}{|z+t|^2} = \frac{1}{8}\int_0^\infty \frac{dt}{(x+t)^2+y^2}.$$

If $x < \delta$ and $|y| \geqslant \delta$, then the last integral on the right, which equals

$$\frac{1}{8|y|}\left(\frac{\pi}{2} - \arctan\frac{x}{|y|}\right),$$

is less than $\pi/8\delta$, while if $x \geqslant \delta$, we have

$$\frac{1}{8}\int_0^\infty \frac{dt}{(x+t)^2+y^2} \leqslant \frac{1}{8}\int_0^\infty \frac{dt}{(x+t)^2} = \frac{1}{8x} \leqslant \frac{1}{8\delta}.$$

[22] The domain G_δ is shown in Figure 10.5.

Thus, in any event,

$$\left|\int_0^\infty \frac{\varphi(t)}{(z+t)^2}\,dt\right| \leqslant \frac{\pi}{8\delta} \tag{10.99}$$

if $z \notin G_\delta$. Applying this estimate to (10.91), after noting that

$$\ln \frac{1}{|\Gamma(z)|} = \operatorname{Re} \ln \frac{1}{\Gamma(z)},$$

we obtain (10.98).

COROLLARY 1. *The entire function* $1/\Gamma(z)$ *is of maximum type.*

Proof. The fact that $1/\Gamma(z)$ is of order 1 has already been noted on p. 307. Applying the estimate (10.98) to a fixed ray $\arg z = \alpha$, where $\pi/2 < |\alpha| < \pi$, we find that

$$\lim_{|z|\to\infty} \frac{\ln \dfrac{1}{|\Gamma(z)|}}{|z| \ln |z|} = -\cos\alpha > 0,$$

i.e., given any $\varepsilon > 0$,

$$\frac{1}{|\Gamma(z)|} > \exp\left[(-\cos\alpha - \varepsilon)|z| \ln |z|\right]$$

for all sufficiently large $|z|$. But then there cannot exist a constant $K > 0$ such that

$$\frac{1}{|\Gamma(z)|} < e^{K|z|}$$

for all sufficiently large $|z|$, and hence $1/\Gamma(z)$ is of maximum type, as asserted.

COROLLARY 2. *Given any* $\varepsilon > 0$,

$$\frac{1}{|\Gamma(z)|} < |z|^{(1+\varepsilon)|z|} \tag{10.100}$$

for all sufficiently large $|z|$.

Proof. Choose a fixed $\delta > 0$ in (10.98) and suppose first that $z \notin G_\delta$. Then

$$\ln \frac{1}{|\Gamma(z)|} = -x(\ln |z| - 1) + y \arg z + \tfrac{1}{2} \ln |z| + C(z) - 1, \tag{10.101}$$

and hence, given any $\varepsilon > 0$,

$$\ln \frac{1}{|\Gamma(z)|} < (1 + \varepsilon)|z| \ln |z| \tag{10.102}$$

for sufficiently large $|z|$. Next suppose that $z \in \overline{G}_\delta$. Then, for sufficiently large $|z|$, $\operatorname{Re} z < -\delta$ and hence $-z \notin G_\delta$.[23] Therefore, for such z, we can use formula (10.98) for the value $-z = \xi + i\eta$, after first noting that

$$\ln \frac{1}{|\Gamma(z)|} = \ln \frac{1}{\pi} + \ln |z| + \ln |\sin \pi z| - \ln \frac{1}{|\Gamma(-z)|},$$

which follows from (10.62), written in the form

$$\frac{1}{\Gamma(z)} = -\frac{1}{\pi} z \sin \pi z \Gamma(-z).$$

The result is

$$\begin{aligned} \ln \frac{1}{|\Gamma(z)|} &= \ln \frac{1}{\pi} + \ln |z| + \ln \sqrt{\sin^2 \pi\xi + \sinh^2 \pi\eta} \\ &\quad + \xi(\ln |z| - 1) - \eta \arg(-z) - \tfrac{1}{2} \ln |z| - C(-z) + 1 \\ &< (1 + \varepsilon)|z| \ln |z|, \end{aligned}$$

where the final inequality is the same as (10.102). Therefore

$$\frac{1}{|\Gamma(z)|} < \exp[(1 + \varepsilon)|z| \ln |z|] \tag{10.103}$$

for all sufficiently large $|z|$, which is equivalent to (10.100).

COROLLARY 3. *If D_ε denotes the domain $|\arg z| < \pi - \varepsilon$ $(0 < \varepsilon < \pi)$, then*

$$\lim_{z \to \infty} \int_0^\infty \frac{\varphi(t)}{(z + t)^2} dt = 0 \qquad (z \in D_\varepsilon). \tag{10.104}$$

Proof. Given any $\delta > 0$, we can find a number $R(\delta, \varepsilon) > 0$, such that if $z \in D_\varepsilon$, $|z| > R(\delta, \varepsilon)$, then $z \notin G_\delta$. This is an immediate consequence of the fact that $G_\delta \cap D_\varepsilon$ is a bounded set (see Figure 10.5). Therefore the inequality (10.99) holds if $z \in D_\varepsilon$, $|z| > R(\delta, \varepsilon)$, and the corollary follows, since δ can be chosen arbitrarily large.

THEOREM 10.16. *Equation* (10.91) *can be written in the form*

$$\ln \frac{1}{\Gamma(z)} = -(z - \tfrac{1}{2}) \ln z + z - \tfrac{1}{2} \ln 2\pi + \int_0^\infty \frac{\varphi(t)}{(z + t)^2} dt + 2\pi i k(z). \tag{10.105}$$

Proof. According to (10.62),

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -\frac{z \sin \pi z}{\pi},$$

and hence

$$\frac{1}{\Gamma(iy)\Gamma(-iy)} = \frac{y \sinh \pi y}{\pi}.$$

---

[23] In fact, $z \in \overline{G}_\delta$, $\operatorname{Re} z < -\delta$ implies that $-z = \xi + i\eta$ belongs to the half-strip $\xi > \delta$, $|\eta| \leqslant \delta$.

But $\Gamma(-iy) = \overline{\Gamma(iy)}$, since $\Gamma(z)$ takes real values for real z, as can be seen, for example, from (10.60), and therefore

$$\frac{1}{|\Gamma(iy)|^2} = \frac{y \sinh \pi y}{\pi}.$$

Setting $z = iy$, $y > 0$ in (10.91), and taking the real part of the resulting equation, we obtain

$$\begin{aligned}\frac{1}{2} \ln \frac{1}{|\Gamma(iy)|^2} &= \frac{1}{2} \ln \frac{y \sinh \pi y}{\pi} \\ &= -1 + \frac{1}{2} \ln y + \frac{\pi}{2} y + \operatorname{Re} \int_0^\infty \frac{\varphi(t)}{(iy + t)^2}\, dt - \int_0^\infty \frac{\varphi(t)}{(1 + t)^2}\, dt\end{aligned}$$

or

$$\int_0^\infty \frac{\varphi(t)}{(1 + t)^2}\, dt = \frac{1}{2} \ln 2\pi - 1 - \frac{1}{2} \ln \frac{e^{\pi y} - e^{-\pi y}}{e^{\pi y}} + \operatorname{Re} \int_0^\infty \frac{\varphi(t)}{(iy + t)^2}\, dt. \tag{10.106}$$

According to (10.104),

$$\lim_{y \to \infty} \operatorname{Re} \int_0^\infty \frac{\varphi(t)}{(iy + t)^2}\, dt = 0,$$

if we choose $\varepsilon < \pi/2$. Thus, passing to the limit $y \to \infty$ in (10.106), we find that

$$\int_0^\infty \frac{\varphi(t)}{(1 + t)^2}\, dt = \tfrac{1}{2} \ln 2\pi - 1.$$

Substitution of this expression into (10.91) gives (10.105), as asserted.

COROLLARY. *The gamma function has the following asymptotic representation, known as Stirling's formula*:

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z - \frac{1}{2}} e^{-z} \qquad (z \in D_\varepsilon). \tag{10.107}$$

Proof. Equation (10.105) is equivalent to

$$\Gamma(z) = \sqrt{2\pi}\, z^{z - \frac{1}{2}} \exp\left[-z - \int_0^\infty \frac{\varphi(t)}{(z + t)^2}\, dt\right].$$

But, according to (10.104),

$$\lim_{z \to \infty} \frac{\Gamma(z)}{\sqrt{2\pi}\, z^{z - \frac{1}{2}} e^{-z}} = \lim_{z \to \infty} \exp\left[-\int_0^\infty \frac{\varphi(t)}{(z + t)^2}\, dt\right] = 1 \qquad (z \in D_\varepsilon),$$

and this is just what is meant by the sign $\sim$ in (10.107).

Remark. For $z = n + 1$, (10.107) becomes

$$n! \sim \sqrt{2\pi}\, (n + 1)^{n + \frac{1}{2}} e^{-n - 1}$$

or

$$n! \sim \sqrt{2\pi}\, n^{n + \frac{1}{2}} e^{-n} = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \tag{10.108}$$

since $\Gamma(n+1) = n!$ and

$$(n+1)^{n+\frac{1}{2}} = n^{n+\frac{1}{2}}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{1/2} \sim en^{n+\frac{1}{2}}.$$

PROBLEMS

10.1. Find two sequences $\{\zeta_n\}$ with the same convergence exponent $\tau > 0$, such that the series

$$\sum_{n=1}^{\infty} \frac{1}{|\zeta_n|^{\tau}}$$

converges in one case and diverges in the other.

10.2. Given a convergent series

$$\sum_{n=1}^{\infty} a_n$$

such that $a_n \geqslant a_{n+1} > 0$ $(n = 1, 2, \ldots)$, prove that

$$\lim_{n\to\infty} na_n = 0.$$

Hint. For any $\varepsilon > 0$, there is an integer $N(\varepsilon) > 0$ such that

$$a_{n+1} + \cdots + a_{n+p} < \frac{\varepsilon}{2}$$

for all $n > N(\varepsilon)$ and $p > 0$. First set $p = n$, and then $p = n + 1$.

10.3. Find an infinite product expansion of each of the following entire functions:

a) $\cos z$; b) $\cosh z$; c) $\sinh z$; d) $e^z - 1$; e) $e^{az} - e^{bz}$.

Ans.

b) $\prod_{n=0}^{\infty}\left\{1 + \left[\frac{2z}{(2n+1)\pi}\right]^2\right\}$; d) $ze^{z/2}\prod_{n=1}^{\infty}\left\{1 + \frac{z^2}{4n^2\pi^2}\right\}$.

10.4. Derive *Wallis' product*

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1}\cdots.$$

10.5. Show that

$$\frac{\sin(a-z)}{\sin a} = e^{-z\cot a}\prod_{n=-\infty}^{\infty}\left(1 - \frac{z}{a+n\pi}\right)e^{z/(a+n\pi)},$$

if a is not a multiple of π.

Hint. Recall (2.51).

10.6. Deduce the periodicity of $\sin z$ from its infinite product expansion (10.24).

Hint. Note that

$$\frac{\sin \pi z}{\pi} = \lim_{n\to\infty} \varphi_n(z),$$

where

$$\varphi_n(z) = \left(1 - \frac{z}{n}\right)\cdots(1 - z)z(1 + z)\cdots\left(1 + \frac{z}{n}\right).$$

Then show that

$$\varphi_n(z + 1) = -\frac{n + z + 1}{n - z}\varphi_n(z).$$

10.7. Prove that if $f(z)$ is an entire function of genus 0 or 1, and if $f(z)$ is real for real z, then all the zeros of $f'(z)$ are real. Give an example showing that the result cannot be extended to functions of genus 2.

10.8. Prove that no entire function of order ρ can have more than one Borel exceptional value.

Hint. We already know that ρ must be an integer (see Theorem 10.7). Suppose there are two Borel exceptional values A and B, and let the sequences of A-points and B-points have convergence exponents $\tau_A < \rho$ and $\tau_B < \rho$, respectively. Then, according to Hadamard's factorization theorem,

$$f(z) - A = g_A(z)e^{Q_A(z)}, \quad f(z) - B = g_B(z)e^{Q_B(z)},$$

where $Q_A(z)$ and $Q_B(z)$ are polynomials of degree ρ, and $g_A(z)$ and $g_B(z)$ are canonical products of the form

$$g_A(z) = z^{\lambda_A}\prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right)\exp\left(\frac{z}{a_n} + \cdots + \frac{z^{\varkappa}}{\varkappa a_n^{\varkappa}}\right),$$

$$g_B(z) = z^{\lambda_B}\prod_{n=1}^{\infty}\left(1 - \frac{z}{b_n}\right)\exp\left(\frac{z}{b_n} + \cdots + \frac{z^{\mu}}{\mu b_n^{\mu}}\right),$$

representing functions of order less than ρ (according to Borel's theorem). It follows that

$$g_A(z)e^{Q_A(z) - Q_B(z)} = g_B(z) + (B - A)e^{-Q_B(z)}, \tag{10.109}$$

which implies that $Q_A(z) - Q_B(z)$ is of degree ρ. Differentiation of (10.109) gives

$$[g_A'(z) + g_A(z)Q_A'(z)]e^{Q_A(z)} = [g_B'(z) + g_B(z)Q_B'(z)]e^{Q_B(z)},$$

and then using the result of Problem 9.8, we find that the coefficient of $e^{Q_A(z)}$ is of order less than ρ, and similarly for the coefficient of $e^{Q_B(z)}$. Now use Hadamard's factorization theorem again to show that $Q_A(z) - Q_B(z)$ is of order less than ρ, contrary to what was just shown.

10.9. Derive the results of Examples 2 and 3, Sec. 51, directly from Mittag-Leffler's theorem.

Hint. In Example 2, write

$$f(z) = \varphi(z) + \sum_{n=1}^{\infty}\left[\frac{1}{z - \zeta_n} + P_n(z)\right],$$

and in Example 3,

$$f(z) = \varphi(z) + \sum_{n=1}^{\infty}\left[\frac{1}{(z - \zeta_n)^2} + P_n(z)\right],$$

where in each case $\varphi(z)$ is entire. Then determine polynomials $P_n(z)$ which guarantee uniform convergence of the series on every compact set containing none of the points ζ_n.

10.10. Generalize Examples 1–3 of Sec. 51 to the case where the sequence $\{\zeta_n\}$ contains the number zero.

10.11. Deduce Weierstrass' theorem from Mittag-Leffler's theorem.

10.12. Generalize Mittag-Leffler's theorem to the case of functions with essential singular points as well as poles.

10.13. Let $\psi(z)$ be the logarithmic derivative of the gamma function, i.e.,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Prove that

a) $\psi(z+1) - \psi(z) = \frac{1}{z}$; b) $\psi(1-z) - \psi(z) = \pi \cot \pi z$.

10.14. Prove that the gamma function satisfies the following relation, known as the *duplication formula*:

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}).$$

Hint. Starting from (10.60), show that

$$\psi'(z) + \psi'(z+\tfrac{1}{2}) = 2\psi'(2z), \tag{10.110}$$

where $\psi(z)$ is the same as in the preceding problem. Then integrate (10.110)

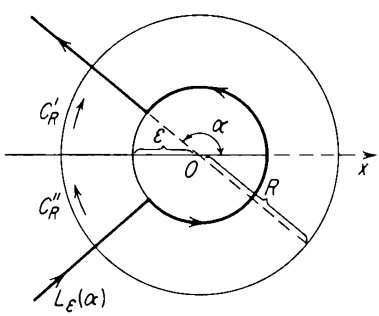

FIGURE 10.6

10.15. Let α be any number in the interval $(\pi/2, \pi)$. Prove that the contour of integration L_ε in formulas (10.80) and (10.85) can be replaced by the contour $L_\varepsilon(\alpha)$ obtained by traversing the infinite line segment $\arg t = -\alpha$, $|t| \geqslant \varepsilon$ from infinity to $\varepsilon e^{-i\alpha}$, the arc of the circle $|z| = \varepsilon$ starting at $\varepsilon e^{-i\alpha}$ and ending at $\varepsilon e^{i\alpha}$, and finally the segment $\arg t = \alpha$, $|t| \geqslant \varepsilon$ from $\varepsilon e^{i\alpha}$ to infinity.

Hint. Show that the integral

$$\frac{1}{2\pi i}\int e^t t^{-z}\, dt$$

evaluated along each of the arcs C_R' and C_R'' shown in Figure 10.6 goes to zero as $R \to \infty$, since on these arcs

$$|e^t t^{-z}| = \exp(R\cos\theta - x\ln R + y\theta)$$
$$\leqslant \exp[-R\cos(\pi - \alpha) - x\ln R + 2\pi|y|].$$

10.16. Show that the estimate (10.103) can be replaced by the sharper estimate

$$\frac{1}{|\Gamma(z)|} < \exp\left[|z| \ln \frac{|z|}{e} + \frac{\pi^2|z|}{\ln |z|}(1 + \varepsilon)\right],$$

where $\varepsilon > 0$ is arbitrary and $|z|$ is sufficiently large.

Hint. Write

$$\ln \frac{1}{|\Gamma(\rho e^{i\varphi})|} = -\rho(\ln \rho - 1)\cos \varphi + \rho\varphi \sin \varphi + \tfrac{1}{2} \ln \rho + C(\rho e^{i\varphi}) - 1$$

[cf. (10.101)], and maximize the function $-\rho(\ln \rho - 1)\cos \varphi + \rho\varphi \sin \varphi$ in the interval $(-\pi, \pi)$.

10.17. Find the order and type of the entire function

$$f(z) = \sum_{n=0}^{\infty} \frac{(Az)^n}{\Gamma(\alpha n + 1)} \qquad (A > 0, \alpha > 0).$$

10.18. Prove that

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt \qquad (\mathrm{Re}\, z > 1),$$

where $\zeta(z)$ is the Riemann zeta function, defined in Problem 1.15.

10.19. Let L_ε^* be the reflection in the origin of the contour figuring in Theorem 10.14, and suppose that $\varepsilon < 2\pi$. Using the result of the preceding problem, and recalling Problem I.18.1, prove that $\zeta(z)$ is a meromorphic function, with a single pole of order 1 at the point $z = 1$.

Hint. The proof closely resembles that of Theorem 10.14.

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